Chapter 0: Introduction

0.3 Assume, for $n \geq 1$, that $H_i(S^n) = \mathbf{Z}$ if $i = 0, n$, and that $H_i(S^n) = 0$ otherwise. Then the equator of the n-sphere is not a retract for $n \geq 1$.

PROOF. First, suppose $n = 1$. Then S^n is the unit circle and $S^{n-1} = \{\pm 1\}$. If there were a continuous map $r : S^n \to S^{n-1}$, then S^{n-1} would be the continuous image of a connected space, making S^{n-1} connected. This is a contradiction.

Now suppose $n > 1$. Suppose, for contradiction, that $r : S^n \to S^{n-1}$ is a retraction. This makes the following diagram commutative:

$$\begin{array}{ccccc} & & S^n & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle r} & \\ S^{n-1} & & \xrightarrow{\quad 1 \quad} & & S^{n-1} \end{array}$$

where i is inclusion and 1 is the identity. It follows from the properties of H_{n-1} that the diagram

$$\begin{array}{ccccc} & & H_{n-1}(S^n) & & \\ & {\scriptstyle H_{n-1}(i)}\nearrow & & \searrow{\scriptstyle H_{n-1}(r)} & \\ H_{n-1}(S^{n-1}) & & \xrightarrow{\quad H_{n-1}(1) \quad} & & H_{n-1}(S^{n-1}) \end{array}$$

is commutative. Note that $H_{n-1}(1)$ is the identity homomorphism. By assumption, $H_{n-1}(S^n) = 0$ and $H_{n-1}(S^{n-1}) = \mathbf{Z}$. This implies $H_{n-1}(r)$ is the trivial homomorphism, which is a contradiction since the diagram commutes. $\square$

0.4 If X is a topological space homeomorphic to D^n, then every continuous $f : X \to X$ has a fixed point

PROOF. Let $h : D^n \to X$ be a homeomorphism. Let $g : D^n \to D^n$ be defined by $g = h^{-1} \circ f \circ h$. Then g is continuous, so that g has a fixed point x by the Brouwer fixed-point theorem. Thus $g(x) = x$, so that $f(h(x)) = h(x)$. It follows that $h(x)$ is a fixed point of f. □

0.10. EXAMPLE. Let G be a **monoid**, that is, a semigroup with 1. Then the following construction gives a category $\mathscr{C}$. Let obj $\mathscr{C}$ have exactly one element, denoted by $*$; define $\mathrm{Hom}(*, *) = G$, and define composition $G \times G \to G$ as the given multiplication in G. Since multiplication in G is well-defined, it follows that composition is well-defined. We check axioms (i), (ii), and (iii). (i) is true vacuously. (ii) is true since multiplication is associative in G. The identity 1 in G is the identity $1_* \in \mathrm{Hom}(*, *)$. Thus $\mathscr{C}$ is a category. Note that this example shows morphisms may not be functions.

0.12. EXAMPLE. Given a category $\mathscr{C}$, we show that the following construction gives a category $\mathscr{M}$. First, an object of $\mathscr{M}$ is a morphism of $\mathscr{C}$, Next if $f, g \in \operatorname{obj}\mathscr{M}$, say $f : A \to B$ and $g : C \to D$, then a morphism of $M : f \to g$ in $\mathscr{M}$ is an ordered pair (h, k) of morphisms of $\mathscr{C}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & D \end{array}$$

commutes. Define composition coordinatewise:

$$(h', k') \circ (h, k) = (h' \circ h, k' \circ k).$$

Let $f : A \to B$, $f' : C \to D$, and $f'' : E \to F$ be objects in $\mathscr{M}$. Let $(h, k) \in \operatorname{Hom}(f, f')$ and $(h', k') \in \operatorname{Hom}(f', f'')$. Then $(h' \circ h, k' \circ k) \in \operatorname{Hom}(f, f'')$ since $f'' \circ h' \circ h = k' \circ f' \circ h = k' \circ k \circ f$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{f'} & D \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle k'} \\ E & \xrightarrow{f''} & F \end{array}$$

Hence, composition is well-defined. To check axiom (ii), observe that composition is associative since corrdinatewise composition is associative. Finally, for each $f : A \to B$, we have $1_f = (1_A, 1_B)$.

0.13. EXAMPLE. We show thaqt $\mathbf{Top^2}$ is a subcategory of $\mathscr{M}$, where $\mathscr{M}$ is the category constructed in Exercise 0.12 with $\mathscr{C} = \mathbf{Top}$. We regard a pair $(X, A) \in$ obj $\mathbf{Top^2}$ as an inclusion $i : A \hookrightarrow X$. Since $i \in \mathrm{Hom}_{\mathscr{C}}(A, X)$, we have $i \in \mathrm{obj}\mathscr{M}$. Now let $B \subset Y$, let $j : B \hookrightarrow Y$ be inclusion, and let $(f, f') \in \mathrm{Hom}_{\mathbf{Top^2}}(i, j)$. Then the following diagram commutes:

$$\begin{array}{ccc} A & \stackrel{i}{\hookrightarrow} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B & \stackrel{j}{\hookrightarrow} & Y \end{array}$$

Thus $(f, f') \in \mathrm{Hom}_{\mathscr{M}}(i, j)$. Finally, composition is coordinatewise in both $\mathbf{Top^2}$ and $\mathscr{M}$.

0.14 EXAMPLE. Let G be a group and let $\mathscr{C}$ be the one object category it defines (Exercise 0.10 applies because every group is a monoid): $\mathrm{obj}\mathscr{C} = \{*\}$, $\mathrm{Hom}(*, *) = G$, and composition is the group operation. If H is a normal subgroup of G, define $x \sim y$ to mean $xy^{-1} \in H$. We show that $\sim$ is a congruence on $\mathscr{C}$ and that $[*, *] = G/H$ is the corresponding quotient category. We know that $\sim$ is an equivalence relation on $G = \mathrm{Hom}(*, *)$ from abstract algebra. Property (i) for congruences is satisfied trivially. For property (ii), suppose $a(a')^{-1} \in H$ and $b(b')^{-1} \in H$. Since H is normal, we have

$$(ab)(a'b')^{-1} = ab(b')^{-1}(a')^{-1} = ah(a')^{-1} = h'a(a')^{-1} = h'h'' \in H,$$

where $h, h', h'' \in H$, and so $ab \sim a'b'$. For the corresponding quotient category, observe that $\mathrm{obj}\mathscr{C}' = \mathrm{obj}\mathscr{C} = \{*\}$, $\mathrm{Hom}_{\mathscr{C}'}(*, *) = \{aH : a \in H\}$, and $aH \circ bH = abH$. Thus $[*, *] = G/H$.

0.17. Let $\mathscr{C}$ and $\mathscr{A}$ be categories, and let $\sim$ be a congruence on $\mathscr{C}$. If $T : \mathscr{C} \longrightarrow \mathscr{A}$ is a functor, then T defines a functor $T' : \mathscr{C}' \longrightarrow \mathscr{A}$ (where $\mathscr{C}'$ is the quotient category) by $T'(X) = T(X)$ for every object X ad $T'([f]) = T(f)$ for every morphism f.

PROOF. We verify that T satisfies properties (i)-(iv) in the definition of functor. For property (i), let $X \in \mathrm{obj}\mathscr{C}'$. Then $X \in \mathrm{obj}\mathscr{C}$ and $T'(X) = T(X) \in \mathrm{obj}\mathscr{A}$. For (ii), let $[f]$ be a morphism in $\mathscr{C}'$. Then $T'([f]) = T(f)$ is a morphism in $\mathscr{A}$ since T is a functor. For (iii), let $[f], [g]$ be morphisms in $\mathscr{C}'$ for which $[g] \circ [f]$ is defined. Then

$$\begin{aligned} T'([g] \circ [f]) &= T'([g \circ f]) = T(g \circ f) \\ &= T(g) \circ T(f) = T'([g]) \circ T'([f]). \end{aligned}$$

For (iv), if $A \in \mathrm{obj}\mathscr{C}'$, then $A \in \mathrm{obj}\mathscr{C}$ and

$$T'([1_A]) = T(1_A) = 1_{TA} = 1_{T'A},$$

since T is a functor and $TA = T'A$. □

0.20. EXAMPLE. If X is a topological space, let $C(X)$ be the set of all continuous real-valued functions on X. We show that $X \mapsto C(X)$ gives a (contravariant) functor **Top** $\longrightarrow$ **Rings**. It is clear that $C(X)$ is a commutative ring with 1. Let $m : X_1 \longrightarrow X_2$ be continuous. Define $Cm : C(X_2) \longrightarrow C(X_1)$ by $Cm(f) = f \circ m$. Then

$$Cm(f + g) = (f + g) \circ m = (f \circ m) + (g \circ m)$$

and

$$Cm(fg) = (fg) \circ m = (f \circ m)(g \circ m).$$

Hence, Cm is a homomorphism. To check property (iii), observe that if $n : X_2 \longrightarrow X_3$ is continuous, then

$$C(n \circ m)(f) = f \circ n \circ m = (Cm \circ Cn)(f).$$

Finally, $C1_X(f) = f \circ 1_X = f$, so that $C1_X$ is the identity map on $C(X)$; that is, $C1_X = 1_{CX}$.

1.2. EXAMPLE. **(i)** Suppose $X \approx Y$ and X is contractible. Since X is contractible, there is a continuous map $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$, where $x_0 \in X$, for all $x \in X$. Let $g : Y \to X$ be a homeomorphism. Define $G : Y \times I \to Y$ by

$$G(y, i) = g^{-1} \circ F(g(y), i).$$

Then G is continuous since g, g^{-1}, and F are. Observe that $G(y, 0) = g^{-1}(g(y)) = y$ and $G(y, 1) = g^{-1}(x_0)$ for all $y \in Y$. It follows that 1_Y is null-homotopic. Thus Y is contractible.

(ii) If X and Y are subspaces of euclidean space, $X \approx Y$, and X is convex, then Y might not be convex. For example, a line segment and the closed half circle are homeomorphic. However, the line segment is convex and the closed half circle is not.

1.3. Let $R : S^1 \to S^1$ be rotation by α radians. Then $R \simeq 1_S$, where 1_S is the identity map of S^1. Thus every continuous map $f : S^1 \to S^1$ is homotopic to a continuous map $g : S^1 \to S^1$ with $g(1) = 1$ (where $1 = e^{2\pi i 0} \in S^1$).

PROOF. Define $R_\alpha : S^1 \to S^1$ by $R_\alpha(e^{it}) = e^{i(t+\alpha)}$. Define $F : S^1 \times I \to S^1$ by

$$F(e^{it}, s) = e^{i(t+\alpha(1-s))}.$$

Then F is continuous since small perturbations in S^1 and s produce small perturbations in the image. Observe that $F(e^{it}, 0) = R_\alpha(e^{it})$ and $F(e^{it}, 1) = e^{it}$. Therefore, $F : R_\alpha \simeq 1_S$.

Now suppose $f : S^1 \to S^1$ is a continuous map and suppose $f(1) = e^{i\beta}$. Take $g = R_{-\beta} \circ f$. Then $g \simeq 1_S \circ f = f$ by Theorem 1.3. □

1.13. For a fixed t with $0 \le t < 1$, $x \mapsto [x, t]$ defines a homeomorphism from a space X to a subspace of CX.

PROOF. For $A \subset X$, define $\overline{A \times \{t\}} = \{[x, t] : x \in A\}$. It is easy to see that $\{\overline{U \times \{t\}} : U \text{ is open in } X\}$ is a basis for $\overline{X \times \{t\}}$. Define $g : X \to \overline{X \times \{t\}}$ by $g(x) = [x, t]$. Clearly, g is bijective. If U is open in X, we have $g(U) = \overline{U \times \{t\}}$. Therefore, g and g^{-1} are continuous. Thus g is a homeomorphism. □

1.29. EXAMPLE. For $n \geq 1$, we show that S^n is a deformation retract of $\mathbf{R}^{n+1} - \{0\}$. Let $X = \mathbf{R}^{n+1} - \{0\}$, and define $F : X \times I \longrightarrow X$ by

$$F(x,t) = (1-t)x + tx/\|x\|.$$

THen F is continuous and $r : X \longrightarrow S^n$, defined by $r(x) = F(x,1)$ is continuous. Since $F(x,0) = x$ for all $x \in X$, $F(x,1) = r(x) \in S^n$ for all $x \in X$ and $F(x,1) = x$ for all $x \in S^n$, it follows that S^n is a deformation retract of X.

1.31. Let $a = (0,\ldots,0,1)$ and $b = (0,\ldots,0,-1)$ be the north and south poles, respectively, of S^n. Then the equator, S^{n-1} is a deformation retract of $S^n - \{a,b\}$; hence, S^{n-1} and $S^n - \{a,b\}$ have the same homotopy type.
PROOF. Let $X = S^n - \{a,b\}$ and $Y = \mathbf{R}^n - \{0\}$. Let $g : X \longrightarrow Y$ be the homeomorphism induced by stereographic projection. By Exercise 1.29, S^{n-1} is a deformation retract of Y. Hence, there is a continuous function $G : Y \times I \longrightarrow Y$ such that $G(y,0) = y$ for all $y \in Y$, $G(y,1) \in S^{n-1}$ for all $y \in Y$, and $G(y,1) = y$ for all $y \in S^{n-1}$. Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = g^{-1} \circ G(g(x),t).$$

Then F is continuous. Observe that $F(x,0) = x$ for all $x \in X$, $F(x,1) \in S^{n-1}$ for all $x \in X$, and $F(x,1) = x$ for all $x \in S^{n-1}$ since $g : x \mapsto x$ for all $x \in S^{n-1}$. It follows that S^{n-1} is a deformation retract of X. $\square$

Chapter 3: The Fundamental Group

3.1. Generalize Theorem 1.3 as follows. Let $A \subset X$ and $B \subset Y$ be given. Assume that $f_0, f_1 \colon X \longrightarrow Y$ with $f_0|A = f_1|A$ and $f_i(A) \subset B$ for $i = 0, 1$; assume that $g_0, g_1 \colon Y \longrightarrow Z$ with $g_0|B = g_1|B$. If $f_0 \simeq f_1$ rel A and $g_0 \simeq g_1$ rel B, then $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel A.
PROOF. Let $F : f_0 \simeq f_1$ rel A and $G : g_0 \simeq g_1$ rel B be homotopies. Define $H : X \times I \longrightarrow Z$ by $H(x,t) = G(f_0(x), t)$. Clearly, $H : g_0 \circ f_0 \simeq g_1 \circ f_0$ rel $\dot{\mathrm{I}}$. Next, let $K : X \times I \longrightarrow Z$ be the composite $g_1 \circ F$. Then $K : g_1 \circ f_0 \simeq g_1 \circ f_1$ rel $\dot{\mathrm{I}}$. This, together with the transitivity of the relative homotopy relation implies our result. □

3.2 **(i)** If $f : \mathrm{I} \longrightarrow X$ is a path with $f(0) = f(1) = x_0 \in X$, then there is a continuous $f' : S^1 \longrightarrow X$ given by $f'(e^{2\pi i t}) = f(t)$. If $f, g : \mathrm{I} \longrightarrow X$ are paths with $f(0) = f(1) = x_0 = g(0) = g(1)$ and if $f \simeq g$ rel $\dot{\mathrm{I}}$,then $f' \simeq g'$ rel $\{1\}$.
(ii) If f and g are as above, then $f \simeq f_1$ rel $\dot{\mathrm{I}}$ and $g \simeq g_1$ rel $\dot{\mathrm{I}}$ implies that $f' * g' \simeq f_1' * g_1'$ rel $\{1\}$
PROOF. **(i)** First, f' is well-defined and continuous since $e^{2\pi i t} \longrightarrow [t]$ is a homeomorphism S^1 to $\mathrm{I}/\dot{\mathrm{I}}$. Suppose $F : f \simeq g$ rel $\dot{\mathrm{I}}$. Define $F' : S^1 \times \mathrm{I} \longrightarrow X$ by $F'(e^{2\pi i t}, s) = F(t, s)$. Then $F' : f' \simeq g'$ rel $\{1\}$.
(ii) By Theorem 3.1, there is a homotopy $H : f * g \simeq f_1 * g_1$ rel $\dot{\mathrm{I}}$. Define $H' : S^1 \times \mathrm{I} \longrightarrow X$ by $H'(e^{2\pi i t}, s) = H(t, s)$. Then $H' : f' * g' \simeq f_1' * g_1'$ rel $\{1\}$. □

3.4 Let $\sigma : \triangle^2 \longrightarrow X$ be continuous, where $\triangle^2 = [e_0, e_1, e_2]$. Define $\varepsilon_0 : I \longrightarrow \triangle^2$ as the affine map with $\varepsilon_0(0) = e_1$ and $\varepsilon_1(1) = e_2$; similarly, define ε_1 by $\varepsilon_1(0) = e_0$ and $\varepsilon_0(1) = e_2$, and define ε_2 by $\varepsilon_2(0) = e_0$ and $\varepsilon_2(1) = e_1$. Finally, define $\sigma_i = \sigma \circ \varepsilon_i$ for $i = 0, 1, 2$.
(i) $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ is nullhomotopic rel $\dot{\text{I}}$.
(ii) $(\sigma_1 * \sigma_0^{-1}) * \sigma_2^{-1}$ is nullhomotopic rel $\dot{\text{I}}$.
(iii) Let $F : \text{I}\times\text{I} \longrightarrow X$ be continuous, and define paths $\alpha, \beta, \gamma, \delta$ in X as located in the figure. Thus $\alpha(t) = F(t, 0)$, $\beta(t) = F(t, 1)$, $\gamma(t) = F(t, 0)$, and $\delta(t) = F(1, t)$. Then $\alpha \simeq \gamma * \beta * \delta^{-1}$ rel $\dot{\text{I}}$.

3.6. **(i)** If $f \simeq g$ rel $\dot{\mathrm{I}}$, then $f^{-1} \simeq g^{-1}$ rel $\dot{\mathrm{I}}$, where f, g are paths in X.
(ii) If f and g are paths in X with $\omega(f) = \alpha(g)$, then

$$(f * g)^{-1} = g^{-1} * f^{-1}.$$

(iii) There exists a closed path f with $f * f^{-1} \neq f^{-1} * f$.
(iv) If $\alpha(f) = p$ and f is not constant, then $i_p * f \neq f$.
PROOF. **(i)** Let $F : f \simeq g$ rel $\dot{\mathrm{I}}$. Define $F' : \mathrm{I} \times \mathrm{I} \longrightarrow X$ by $F'(t,s) = F(1-t,s)$. Then F is continuous. Moreover, $F'(t,0) = f^{-1}(t)$ and $F'(t,1) = g^{-1}(t)$ for all $t \in \mathrm{I}$. Finally, $F'(0,s) = f^{-1}(0) = g^{-1}(0)$ and $F'(1,s) = f^{-1}(1) = g^{-1}(1)$ for all $s \in \mathrm{I}$. Therefore, $F' : f^{-1} \simeq g^{-1}$ rel $\dot{\mathrm{I}}$.
(ii) We have

$$\begin{aligned}
(f * g)^{-1}(t) &= (f * g)(1-t) \\
&= \begin{cases} f(2(1-t)) & \text{if } 0 \leq 1-t \leq 1/2 \\ g(2(1-t)-1) & \text{if } 1/2 \leq 1-t \leq 1 \end{cases} \\
&= \begin{cases} g(1-2t) & \text{if } 0 \leq t \leq 1/2 \\ f(1-(2t-1)) & \text{if } 1/2 \leq t \leq 1 \end{cases} \\
&= \begin{cases} g^{-1}(2t) & \text{if } 0 \leq t \leq 1/2 \\ f^{-1}(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases} \\
&= (g^{-1} * f^{-1})(t).
\end{aligned}$$

(iii) Define $f : I \longrightarrow S^1$ by $f(t) = e^{2\pi i t}$. Clearly, $f * f^{-1} \neq f^{-1} * f$. For example, $(f * f^{-1})(1/8) = e^{\pi i/2}$ and $(f^{-1} * f)(1/8) = e^{3\pi i/2}$.
(iv) We show that there is $t \in \mathrm{I}$ such that $(i_p * f)(t) \neq f(t)$. If $t \leq 1/2$ and $f(t) \neq p$, then clearly, $(i_p * f)(t) \neq f(t)$. Therefore, we may assume that $f(t) = p$ for all t in $[0, 1/2]$. Assume, for contradiction, that $(i_p * f)(t) = f(2t-1) = f(t)$ for all $t \in (1/2, 1]$. Let $t \in (1/2, 1)$. Then by repeated applications of the equality $f(t) = f(2t-1)$, we obtain $f(t) = f(s)$ for some $s \in [0, 1/2]$. Thus $f(t) = p$. This is a contradiction since f is not constant. □

3.11 If $X = \{x_0\}$ is a one-point space, then $\pi_1(X, x_0) = \{1\}$.
PROOF. Since X has only one element, there is only one path in X; namely $t \mapsto x_0$ for all $t \in$ I. Thus $\pi_1(X, x_0)$ is a group with one element. □

3.14. If f is a closed path in S^1 at 1 and if $m \in \mathbf{Z}$, then $t \mapsto f(t)^m$ is a closed path in S^1 at 1 and

$$\deg(f^m) = m\deg(f).$$

PROOF. Let $\tilde{f}$ be the lifting of f with $\tilde{f}(0) = 0$ and let $\tilde{f^m}$ be the lifting of f^m with $\tilde{f^m}(0) = 0$. We show that $\tilde{f^m} = m\tilde{f}$. Clearly, $m\tilde{f}$ is continuous and $m\tilde{f}(0) = m \cdot 0 = 0$. Further, $f^m = (\exp\tilde{f})^m = \exp m\tilde{f}$. It follows that

$$\deg(f^m) = \tilde{f^m}(1) = m\tilde{f}(1) = m\deg f.$$

□

3.16. If T is the torus $S^1 \times S^1$, then

$$\pi_1(T, t_0)) \cong \mathbf{Z} \times \mathbf{Z}.$$

PROOF. By Theorem 3.16, $\pi_1(S^1, t_1) \cong \pi_1(S^1, t_2) \cong \mathbf{Z}$ for all t_1, t_2 in S^1. Let $t_0 = (t_1, t_2)$. Then by Theorem 3.7, we have

$$\begin{aligned} \pi_1(T, t_0) &\cong \pi_1(S^1, t_1) \times \pi_1(S^1, t_2) \\ &\cong \mathbf{Z} \times \mathbf{Z}. \end{aligned}$$

□

4.3. For a given space X, define $S_1(X)$ to be the free abelian group with basis all paths $\sigma : \mathrm{I} \to X$, and let $S_0(X)$ be the free abelian group with basis X.
(i) There is a homomorphism $\partial_1 : S_1(X) \to S_0(X)$ with $\partial_1\sigma = \sigma(1) - \sigma(0)$ for every path σ in X.
(ii) If $x_1, x_0 \in X$, then $x_1 - x_0 \in \operatorname{im} \partial_1$ if and only if x_0, x_1 lie in the same path component of X.
(iii) If σ is a path in X, then $\sigma \in \ker \partial_1$ if and only if σ is a closed path. Exhibit a nonzero element of $\ker \delta_1$ that is not a closed path.
PROOF. **(i)** This follows form Theorem 4.1.
(ii) Suppose $x_1 - x_0 \in \operatorname{im} \partial_1$. Then there is a path $\sigma : \mathrm{I} \to X$ with $\partial_1\sigma = \sigma(1) - \sigma(0) = x_1 - x_0$. This implies $\sigma(1) = x_1$ and $\sigma(0) = x_0$. Hence, x_1, x_0 lie in the same path component of X.

Conversely, suppose x_1, x_0 lie in the same path component of X. Then there is a path $\sigma : I \to X$ with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Hence, $x_1 - x_0 = \sigma(1) - \sigma(0) = \partial_1(\sigma)$.
(iii) If σ is a path, we have $\sigma \in \ker \partial_1$ iff $\sigma(1) - \sigma(0) = 0$ iff $\sigma(1) = \sigma(0)$ iff σ is closed. Further, if σ_1, σ_2 are paths in X with $\sigma_1(1) = \sigma_2(0)$ and $\sigma_1(0) = \sigma_2(1)$, then $\sigma_1 + \sigma_2$ is a nonzero element of $\ker \partial_1$ that is not a closed path. □

4.4. EXAMPLE. We show that if $X = \emptyset$, then $H_n(X) = 0$ for all $n \geq 0$. Since $X = \emptyset$, $S_n(X)$ is the free abelian group with basis $\emptyset$, so that $S_n(X) = 0$ by definition (see Thm 4.2). If $n \geq 0$, then $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ is given by $\partial_n : 0 \mapsto 0$. Thus $Z_n(X) = \ker \partial_n = 0$ and $B_n(X) = \operatorname{im} \partial_{n+1} = 0$. Hence, $H_n(X) = Z_n(X)/B_n(X) = 0$.

4.6. For each fixed $n \geq 0$, $S_n : \mathbf{Top} \longrightarrow \mathbf{Ab}$ is a functor.
PROOF. We have already defined $S_n(X)$ on objects X. If $f : X \longrightarrow Y$ is a continuous map, define $f_\# : S_n(X) \longrightarrow S_n(Y)$ by

$$f_\#(\Sigma\, m_\sigma \sigma) = \Sigma\, m_\sigma (f \circ \sigma).$$

Then $S_n(f)$ is a homomorphism since

$$f_\#(\Sigma\, m_\sigma \sigma + \Sigma\, n_\sigma \sigma) = \Sigma\, (m_\sigma + n_\sigma)(f \circ \sigma) = f_\#(\Sigma\, m_\sigma \sigma) + f_\#(\Sigma\, n_\sigma \sigma).$$

To check condition (iii), let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous. Then

$$(g \circ f)_\#(\Sigma\, m_\sigma \sigma) = \Sigma\, m_\sigma (g \circ f \circ \sigma) = (g_\# \circ f_\#)(\Sigma\, m_\sigma \sigma).$$

Finally, if $1_X : X \longrightarrow X$ is the identity, then

$$1_{X\#}(\Sigma\, m_\sigma \sigma) = \Sigma\, m_\sigma 1_X \sigma = \Sigma\, m_\sigma \sigma.$$

□

4.7. EXAMPLE. We show that $H_n(S^0) = 0$ for all $n \geq 0$. by Theorem 4.13,

$$H_n(S^0) \cong H_n(\{-1\}) \times H_n(\{1\}),$$

and by Theorem 4.12, $H_n(\{-1\}) = H_n(\{1\}) = 0$.

4.9. EXAMPLE. Using the explicit formula for β_{n+1},

$$\beta_{n+1} = \sum_{i=0}^{n} (-1)^i [a_0, \ldots, a_i, b_i, b_{i+1}, \ldots, b_n],$$

we show that

$$\partial_{n+1}\, \beta_{n+1} = (\lambda^{\triangle}_{1\#} - \lambda^{\triangle}_{0\#} - P^{\triangle}_{n-1}\, \partial_n)(\delta)$$

for $n = 0$ and $n = 1$. Suppose $n = 0$. Let $a_0 = (e_0, 0)$, and let $b_0 = (e_0, 1)$. Then

$$\partial_1 \beta_1 = \partial_1([a_0, b_0]) = b_0 - a_0.$$

It is easy to see that $\lambda^{\triangle}_{0\#}\delta : e_0 \mapsto (e_0, 0) = a_0$ and $\lambda^{\triangle}_{1\#}\delta : e_0 \mapsto (e_0, 1) = b_0$. Further, $P^{\triangle}_{-1}\partial_0\delta : e_0 \mapsto 0$ by definition of $P^{\triangle}_{-1}$. Thus

$$\begin{aligned}(\lambda^{\triangle}_{1\#} - \lambda^{\triangle}_{0\#} - P^{\triangle}_{-1}\, \partial_0)(\delta) &= \lambda^{\triangle}_{1\#}\, \delta - \lambda^{\triangle}_{0\#}\, \delta - P^{\triangle}_{-1}\partial_0\delta \\ &= b_0 - a_0 \\ &= \partial_1\beta_1.\end{aligned}$$

Now suppose $n = 1$ and let $a_i = (e_i, 0)$, and let $b_i = (e_i, 1)$ for $i = 0, 1$. Then we have

$$\begin{aligned}\partial_2\beta_2 &= \partial_2([a_0, b_0, b_1] - [a_0, a_1, b_1]) \\ &= [b_0, b_1] - [a_0, b_1] + [a_0, b_0] - [a_1, b_1] + [a_0, b_1] - [a_0, a_1] \\ &= [b_0, b_1] + [a_0, b_0] - [a_1, b_1] - [a_0, a_1].\end{aligned}$$

Now $\lambda^{\triangle}_{0\#}\delta : e_i \mapsto (e_i, 0) = a_i$ and $\lambda^{\triangle}_{1\#}\delta : e_i \mapsto (e_i, 1) = b_i$ for $i = 0, 1$. Thus $\lambda^{\triangle}_{0\#}\delta = [a_0, a_1]$ and $\lambda^{\triangle}_{1\#}\delta = [b_0, b_1]$. Observe that

$$\begin{aligned}(P^{\triangle}_0\partial_1)(\delta) &= P^{\triangle}_0(\partial_1\delta) \\ &= P^{\triangle}_0(\varepsilon_0 - \varepsilon_1) \\ &= P^{\triangle}_0(\varepsilon_0) - P^{\triangle}_0(\varepsilon_1) \\ &= (\varepsilon_0 \times 1)_\#(\beta_1) - (\varepsilon_1 \times 1)_\#(\beta_1) \\ &= (\varepsilon_0 \times 1)\beta_1 - (\varepsilon_1 \times 1)\beta_1,\end{aligned}$$

where $\beta_1 = [a_0, b_0]$. Since

$$\begin{aligned}(\varepsilon_0 \times 1)\beta_1 &: e_0 \mapsto a_0 \mapsto (\varepsilon_0 \times 1)(a_0) = a_1, \\ (\varepsilon_0 \times 1)\beta_1 &: e_1 \mapsto b_0 \mapsto (\varepsilon_0 \times 1)(b_0) = b_1, \\ (\varepsilon_1 \times 1)\beta_1 &: e_0 \mapsto a_0 \mapsto (\varepsilon_1 \times 1)(a_0) = a_0, \\ (\varepsilon_1 \times 1)\beta_1 &: e_1 \mapsto b_0 \mapsto (\varepsilon_1 \times 1)(b_0) = b_0,\end{aligned}$$

we have $(\varepsilon_0 \times 1)\beta_1 = [a_1, b_1]$ and $(\varepsilon_1 \times 1)\beta_1 = [a_0, b_0]$. Thus

$$\begin{aligned}(\lambda^{\triangle}_{1\#} - \lambda^{\triangle}_{0\#} - P^{\triangle}_0\, \partial_1)(\delta) &= \lambda^{\triangle}_{1\#}\, \delta - \lambda^{\triangle}_{0\#}\, \delta - P^{\triangle}_0 \partial_1 \delta \\ &= [b_0, b_1] - [a_0, a_1] - ([a_1, b_1] - [a_0, b_0]) \\ &= \partial_2 \beta_2.\end{aligned}$$

An explicit formula for $P^X_1(\sigma)$, where $\sigma : \triangle^1 \to X$ is a 1- simplex, is

$$\begin{aligned}P^X_1(\sigma) &= (\sigma \times 1)_{\#}(\beta_2) \\ &= (\sigma \times 1)_{\#}([a_0, b_0, b_1] - [a_0, a_1, b_1]) \\ &= (\sigma \times 1)[a_0, b_0, b_1] - (\sigma \times 1)[a_0, a_1, b_1].\end{aligned}$$

One thus views $P^X_1(\sigma)$ as the ”triangulated prism over σ.”

4.13. EXAMPLE. The Hurewicz map is "natural." If $h : (X, x_0) \to (Y, y_0)$ is a map of pointed spaces, then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ H_1(X) & \xrightarrow{h_*} & H_1(Y) \end{array}$$

To see this, opbserve that if f is a closed path in X at x_0, then

$$\begin{aligned} (\varphi h_*)([f]) &= \varphi([h \circ f]) = \text{cls } (h \circ f)\eta \\ &= h_*(\text{cls } f\eta) \\ &= (h_*\varphi)([f]) \end{aligned}$$

4.14. If f is a (not necessarily closed) path in X, then the 1-chain f is homologous to $-f^{-1}$.
PROOF. Define a continuous map $\sigma : \triangle^2 \to X$ as indicated in the following picture:

In more detail, first define σ on $\triangle^2$: $\sigma(1-t, t, 0) = f(t)$; $\sigma(0, 1-t, t) = f^{-1}(t)$; $\sigma(1-t, 0, t) = (f * f^{-1})(t)$. Now define σ on all of $\triangle^2$ by setting it constant on the line segments with endpoints $a = a(t) = (1-t, t, 0)$ and $b = b(t) = ((2-t)/2, 0, t/2)$, and constant on the line segments with endpoints $c = c(t) = (0, 1-t, t)$ and $d = d(t) = ((1-t)/2, 0, (1+t)/2)$. It is easy to see that $\sigma : \triangle^2 \to X$ is continuous, that is, $\sigma \in S_2(X)$. Moreover, $\partial\sigma = \sigma\varepsilon_0 - \sigma\varepsilon_1 + \sigma\varepsilon_2 = f^{-1}\eta - (f * f^{-1})\eta + f\eta$. Thus

$$\text{cls}(f^{-1}\eta - (f * f^{-1})\eta + f\eta) = B_1(X),$$

so that

$$\text{cls}(f^{-1}\eta + f\eta) = \text{cls}(f * f^{-1})\eta.$$

But $f * f^{-1}$ is nullhomotopic, so that $\text{cls}(f * f^{-1})\eta = B_1(X)$ by Theorem 4.27. Hence, $\text{cls}(f^{-1}\eta + f\eta) = B_1(X)$, and so, $f\eta$ is homologous to $-f^{-1}\eta$. $\square$

5.1. **(i)** If $0 \to A \xrightarrow{f} B$ is exact, then f is injective (there is no need to label the only possible homomorphism $0 \to A$).
(ii) If $B \xrightarrow{g} C \to 0$ is exact, then g is surjective (there is no need to label the only possible homomorphism $C \to 0$).
(iii) If $0 \to A \xrightarrow{f} B \to 0$ is exact, then f is an isomorphism.
(iv) If $0 \to A \to 0$ is exact, then $A = 0$.

PROOF. **(i)** Suppose $0 \to A \xrightarrow{f} B$ is exact. Then $\ker f = \operatorname{im}(0 \to A) = \{0\}$. Hence, f is injective.
(ii) Suppose $B \xrightarrow{g} C \to 0$ is exact. Then $\operatorname{im} g = \ker(C \to 0) = C$. Hence, g is surjective.
(iii) Suppose $0 \to A \xrightarrow{f} B \to 0$ is exact. Then f is bijective by (i) and (ii). Therefore, f is an isomorphism.
(iv) Suppose $0 \to A \to 0$ is exact. Then

$$0 = \operatorname{im}(0 \to A) = \ker(A \to 0) = A.$$

Thus $A = 0$. □

5.2. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact, then f is surjective if and only if h is injective.
PROOF. Suppose f is surjective. Then $\ker g = \operatorname{im} f = B$, so that

$$\ker h = \operatorname{im} g \cong B/\ker g = B/B \cong 0.$$

Hence, $\ker h = 0$, and so, h is injective. For the converse, suppose h is injective. Then $\operatorname{im} g = \ker h = \{0\}$. It follows that $\operatorname{im} f = \ker g = B$. □

5.3. EXAMPLE. A short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

Then i is injective since $\ker i = \operatorname{im}(0 \to A) = \{0\}$. Hence, $iA \cong A$. Observe that $\ker p = \operatorname{im} i = iA$ and $\operatorname{im} p = \ker(C \to 0) = C$. Therefore, $B/iA \cong C$ via $b + iA \mapsto pb$ by the First Isomorphism Theorem.

5.4. If

$$\cdots \to C_{n+1} \to A_n \xrightarrow{h_n} B_n \to C_n \to A_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \to C_{n-1} \to \cdots$$

is exact and every third arrow $h_n : A_n \to B_n$ is an iosomorphism, then $C_n = 0$ for all n.

PROOF. Since h_n is surjective, it follows that

$$\ker(B_n \to C_n) = \operatorname{im} h_n = B_n.$$

Hence, $\operatorname{im}(B_n \to C_n) = \{0\}$. Since h_{n-1} is injective, it follows that

$$\operatorname{im}(C_n \to A_{n-1}) = \ker h_{n-1} = \{0\}$$

Hence, $\ker(C_n \to A_{n-1}) = C_n$. Since $\operatorname{im}(B_n \to C_n) = \ker(C_n \to A_{n-1})$, it follows that $C_n = 0$. □

5.6. EXAMPLE. If (S_*, ∂) is a complex with $\partial_n = 0$ for every $n \in \mathbf{Z}$, then $H_n(S_*) = S_n$ for every $n \in \mathbf{Z}$:

$$\begin{aligned} H_n(S_*) &= Z_n(S_*)/B_n(S_*) = S_n/\{0\} \\ &\cong S_n. \end{aligned}$$

5.8. A sequence $S'_* \xrightarrow{f} S_* \xrightarrow{g} S''_*$ is exact in **Comp** if and only if $S'_n \xrightarrow{f_n} S_n \xrightarrow{g_n} S''_n$ is exact in **Ab** for every $n \in \mathbf{Z}$.
PROOF. We have im f is

$$\cdots \to \operatorname{im} f_{n+1} \to \operatorname{im} f_n \to \operatorname{im} f_{n-1} \to \cdots,$$

and ker g is

$$\cdots \to \ker g_{n+1} \to \ker g_n \to \ker g_{n-1} \to \cdots,$$

Thus im f = ker g if and only if im $f_n = \ker g_n$ for all n if and only if $S'_n \xrightarrow{f_n} S_n \xrightarrow{g_n} S''_n$ is exact for all n. □

5.11. The third isomorphism theorem holds in **Comp**. If $U_* \subset T_* \subset S_*$ are subcomplexes, then there is a short exact sequence of complexes

$$0 \longrightarrow T_*/U_* \xrightarrow{i} S_*/U_* \xrightarrow{p} S_*/T_* \longrightarrow 0,$$

where $i_n : t_n + U_n \mapsto t_n + U_n$ (inclusion) and $p_n(s_n + U_n) = s_n + T_n$.
PROOF. Let $n \in \mathbf{Z}$. Observe that

$$(\bar{\partial}_n i_n)(t_n + U_n) = \partial_n t_n + U_{n-1} = (i_{n-1}\bar{\partial}_n)(t_n + U_n)$$

and

$$(\bar{\partial}_n p_n)(s_n + U_n) = \partial_n s_n + T_{n-1} = (p_{n-1}\bar{\partial}_n)(s_n + U_n).$$

It follows that i and p are chain maps.

Observe that $\ker i_n = \{U_n\} = \mathrm{im}(0 \to T_n/U_n)$ and $\mathrm{im}\, p_n = S_n/T_n = \ker(S_n/T_n \to 0)$. Finally, we show that $\mathrm{im}\, i_n = \ker p_n$. Let $t_n + U_n \in \mathrm{im}\, i_n$. Then

$$p_n(t_n + U_n) = t_n + T_n = T_n,$$

so that $t_n + U_n \in \ker p_n$. For the reverse inclusion, let $s_n + U_n \in \ker p_n$. Then

$$p_n(s_n + U_n) = s_n + T_n = T_n,$$

so that $s_n \in T_n$. Thus $s_n + U_n \in \mathrm{im}\, i_n$. □

5.12. For every n, $H_n(\Sigma_\lambda S_*^\lambda) \cong \Sigma_\lambda H_n(S_*^\lambda)$.

PROOF. Let $(S_*^\lambda, \partial^\lambda)$, $(\Sigma_\lambda S_*^\lambda, \partial)$ be chain complexes and define $\varphi : H_n(\Sigma_\lambda S_*^\lambda) \to \Sigma_\lambda H_n(S_*^\lambda)$ by $\varphi : \text{cls}(\Sigma_\lambda s_n^\lambda) \mapsto \Sigma_\lambda \text{cls } s_n^\lambda$. Since $\partial_n(\Sigma_\lambda s_n^\lambda) = \Sigma_\lambda \, \partial_n^\lambda s_n^\lambda$, it is easy to see that $\Sigma_\lambda s_n^\lambda \in B_n(\Sigma_\lambda S_*^\lambda)$ if and only if $s_n^\lambda \in B_n(S_*^\lambda)$ for all λ.To show that φ is well-defined, suppose cls $(\Sigma_\lambda s_n^\lambda) = \text{cls}(\Sigma_\lambda t_n^\lambda)$. Then

$$\Sigma(s_n^\lambda - t_n^\lambda) = \Sigma_\lambda s_n^\lambda - \Sigma_\lambda t_n^\lambda \in B_n(\Sigma_\lambda S_*^\lambda),$$

so that $s_n^\lambda - t_n^\lambda \in B_n(S_*^\lambda)$ for all λ. Hence, $\Sigma_\lambda \text{cls } s_n^\lambda = \Sigma_\lambda \text{cls } t_n^\lambda$.

To show that φ is injective, suppose $\Sigma_\lambda \text{cls } s_n^\lambda = \Sigma_\lambda \text{cls } t_n^\lambda$. Then cls $s_n^\lambda =$ cls t_n^λ for all λ, so that $s_n^\lambda - t_n^\lambda \in B_n(S_*^\lambda)$ for all λ. Thus

$$\Sigma_\lambda s_n^\lambda - \Sigma_\lambda t_n^\lambda = \Sigma(s_n^\lambda - t_n^\lambda) \in B_n(\Sigma_\lambda S_*^\lambda),$$

so that $\text{cls}(\Sigma_\lambda s_n^\lambda) = \text{cls}(\Sigma_\lambda t_n^\lambda)$. It is clear that φ is surjective.

Finally, we show that φ is a homomorphism. We have

$$\begin{aligned} \varphi(\text{cls}(\Sigma_\lambda s_n^\lambda) + \text{cls}(\Sigma_\lambda t_n^\lambda)) &= \varphi(\text{cls}(\Sigma_\lambda (s_n^\lambda + t_n^\lambda))) = \Sigma_\lambda \text{cls}(s_n^\lambda + t_n^\lambda) \\ &= \Sigma_\lambda \text{cls} s_n^\lambda + \Sigma_\lambda \text{cls} t_n^\lambda \\ &= \varphi(\text{cls}(\Sigma_\lambda s_n^\lambda)) + \varphi(\text{cls}(\Sigma_\lambda t_n^\lambda)). \end{aligned}$$

□

7.4. **(i)** If K is a simplicial complex and F is a subset of $|K|$, then F is closed if and only if $F \cap s$ is closed in s for every $s \in K$.
(ii) If s is a simplex in K of largest dimension, then $s^\circ = s - \dot{s}$ is an open subset of $|K|$.
PROOF. **(i)** If F is closed in $|K|$, then $F \cap s$ is closed in s by the definition of subspace. For the converse, suppose $F \cap s$ is closed in s for every $s \in K$. Let $s_1, \ldots, s_m$ be the simplexes in K. Then for all j, there is some closed set A_j of $|K|$ such that $F \cap s_j = A_j \cap s_j$. Now each $A_j \cap s_j$ is closed in $|K|$ since each s_j is closed in $|K|$. Thus

$$F = F \cap |K| = F \cap (\bigcup_{j=1}^{m} s_j) = \bigcup_{j=1}^{m} (F \cap s_j)$$
$$= \bigcup_{j=1}^{m} (A_j \cap s_j)$$

is closed in $|K|$.
(ii) Let $s' \in K$. Then $(|K| - s^\circ) \cap s'$ is either $s - s^\circ$ or is all of s'. Hence, $|K| - s^\circ$ is closed in s'. It follows from part (i) that $|K| - s^\circ$ is closed in $|K|$, so that s° is open in $|K|$. □

7.10. A simplicial map $\varphi : K \longrightarrow L$ is a simplicial approximation to $f : |K| \longrightarrow |L|$ if and only if, whenever $x \in |K|$ and $f(x) \in s^\circ$ (where s is a simplex of L), then $|\varphi|(x) \in s$.
PROOF. Suppose $s \in L$ with $f(x) \in s^\circ$. Suppose s' is a simplex in K with $x \in (s')^\circ$. We show that $|\varphi|(s')$ is a face of s. Suppose $|\varphi|(s')$ is not a face of s. Then there is $p \in \text{Vert}(s')$ such that $\varphi(p) \in \text{Vert}(|\varphi|(s')) - \text{Vert}(s)$. But

$$f(x) \in f(\text{st}(p)) \subset \text{st}(\varphi(p)).$$

This is a contradiction since $\varphi(p)$ is not a vertex of s.

For the converse, suppose $p \in \text{Vert}(K)$ and $x \in \text{st}(p)$. The there is $s' \in K$ with $p \in \text{Vert}(s')$ and $x \in (s')^\circ$. Hence,

$$|\varphi|(x) \in |\varphi|((s')^\circ) \subset |\varphi|(s')^\circ.$$

Suppose $f(x) \in s^\circ$ for some $s \in L$. Then $|\varphi|(x) \in s$ by hypothesis. Since $|\varphi|(s')^\circ$ and s have a point in common, namely $|\varphi|(x)$, it follows that $|\varphi|(s')$ is a face of s. Thus $\varphi(p) \in \text{Vert}(s)$, and so,

$$f(x) \in s^\circ \subset \text{st}(\varphi(p)).$$

We have shown that $f(\text{st}(p)) \subset \text{st}(\varphi(p))$. □

7.11. If $\varphi : K \longrightarrow L$ is a simplicial approximation to $f : |K| \longrightarrow |L|$, then $|\varphi| \simeq f$.
PROOF. For each simplex s in K, define $G_s : s \times I \longrightarrow |L|$ by

$$G_s(x, i) = f(x)(1 - i) + |\varphi|(x)i.$$

Note that each G_s is well-defined since if $f(x) \in s^\circ$, then $|\varphi|(x) \in s$ by Exercise 7.10. Thus $G_s : f|s \simeq |\varphi||s$. Thus there is a unique continuous $G : |K| \times I \longrightarrow |L|$ defined by $G(x, i) = G_s(x, i)$ by the Gluing Lemma. Hence, $G : f \simeq |\varphi|$. □