S13 Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subset A$. Then A is open in X.
Proof. For each $x \in A$, let U_x be an open set containing x such that $U_x \subset A$. Then $A = \bigcup_{x \in A} U_x$, so that A is an arbitrary union of open sets. Thus A is open in X. ■

4.
(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, then $\bigcap \mathcal{T}_\alpha$ is a topology on X. Clearly, ϕ, X belong to $\bigcap \mathcal{T}_\alpha$ since ϕ, X belong to $\mathcal{T}_\alpha$ for each α. Next, let $\{U_\beta\}$ be a subcollection of $\bigcap \mathcal{T}_\alpha$. Then for each α, $\{U_\beta\}$ is a subcollection of $\mathcal{T}_\alpha$, so that $\bigcup U_\beta \in \mathcal{T}_\alpha$. Hence, $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$. A similar argument shows that $\bigcap \mathcal{T}_\alpha$ is closed under finite intersections. Thus $\bigcap \mathcal{T}_\alpha$ is a topology on X.

However, it is not necessarily true that $\bigcup \mathcal{T}_\alpha$ is a topology on X. For example, let $X = \{a, b, c\}$ and consider the following two topologies on X:

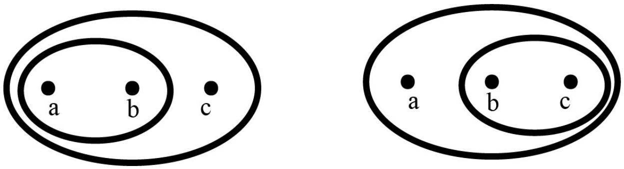

Figure 13.1

Then their union is not a topology on X since $\{a, b\} \cap \{b, c\} = \{b\}$ is not a member of their union.
(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Let $\mathcal{T}$ be the topology on X generated by $\bigcup \mathcal{T}_\alpha$. Then $\mathcal{T}$ is the unique smallest topology on X containing all the $\mathcal{T}_\alpha$. Next, let $\mathcal{T}' = \bigcap \mathcal{T}_\alpha$. Then $\mathcal{T}'$ is the unique largest topology contained in all $\mathcal{T}_\alpha$. Note that $\mathcal{T}'$ is a topology by part (a).
(c) If $X = \{a, b, c\}$, consider the following two topologies on X:

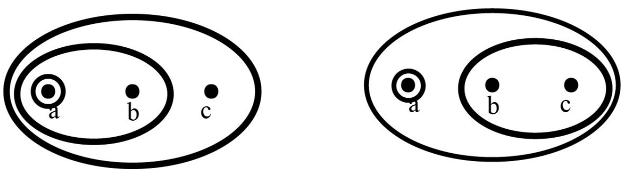

Figure 13.2

Then by part (b), the smallest topology containing these two topologies and the largest topology contained in these two topologies are respectively as follows:

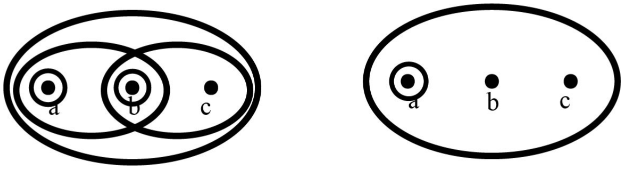

Figure 13.3

6. The topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable.

Proof. Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$, respectively. Given the basis element $[0, x)$ for $\mathcal{T}$ and the point 0 of $[0, x)$, there is no basis element for $\mathcal{T}'$ that contains 0 and lies in $[0, x)$. On the other hand, given the basis element $(-1, 1)-K$ for $\mathcal{T}'$ and the point 0 of $(-1, 1) - K$, there is no basis element for $\mathcal{T}$ that contains 0 and lies in $(-1, 1) - K$. It follows that $\mathcal{T}$ and $\mathcal{T}'$ are not comparable. ■

S16 Exercises

1. If Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology A inherits as a subspace of X.
Proof. Let $\mathcal{T}$ be the topology A inherits as a subspace of X and let $\mathcal{T}'$ be the topology A inherits as a subspace of Y. If U is open in X, then

$$A \cap U = A \cap (Y \cap U).$$

Since a general element of $\mathcal{T}$ is a set of the form $A \cap U$ and a general element of $\mathcal{T}'$ is a set of the form $A \cap (Y \cap U)$, it follows that $\mathcal{T}$ and $\mathcal{T}'$ are the same. ■

4. A map $f: X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. The maps $\pi_1: X \times Y \to X$ and $\pi_2: X \times Y \to Y$ are open maps.
Proof. Let W be an open set of $X \times Y$ and let $x \in \pi_1(W)$. Then there is $x \times y \in W$ such that $\pi_1(x \times y) = x$. Since W is open, there is an open set U of X and an open set V of Y such that $x \times y \in U \times V \subset W$. Then $x \in U \subset \pi_1(W)$. Since every topology is a basis for itself, $\pi_1(W)$ is open. Similarly, π_2 is an open map. ■

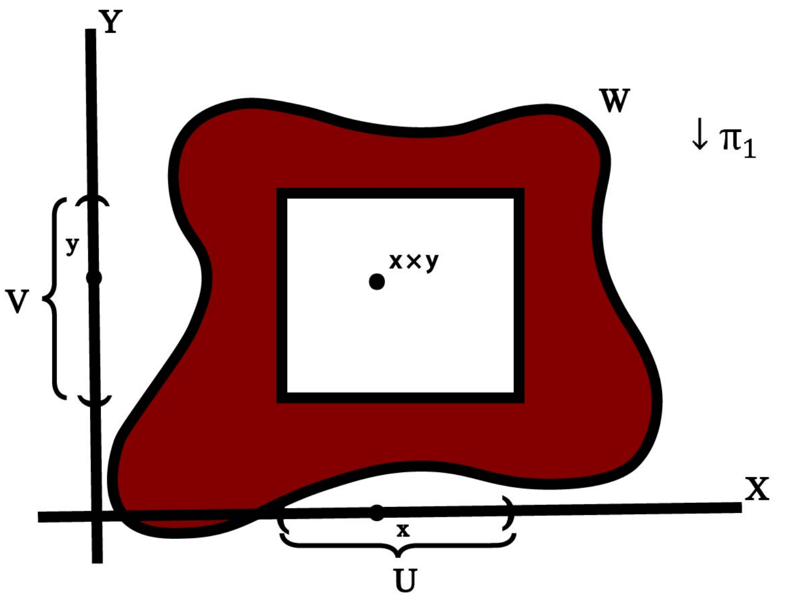

Figure 16.1

6. The countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Proof. We use Theorem 13.1 to show that $\{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$ is a basis for $\mathbb{R}$. The result follows by Theorem 15.1. Let U be an open set of $\mathbb{R}$ and let $x \in U$. Then there is an open interval (a, b) such that $x \in (a, b) \subset U$. Choose rational numbers c, d such that $a < c < x < d < b$. Then $x \in (c, d) \subset U$. ■

9. The dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Proof. Let $\mathcal{T}$ be the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{T}'$ be the product topology on $\mathbb{R}_d \times \mathbb{R}$. Suppose $a, b, c \in \mathbb{R}$ with $b < c$. Then an set of the form $(a \times b, a \times c)$ is a general basis element for $\mathcal{T}$ and a set of the form $\{a\} \times (b, c)$ is a general basis element for $\mathcal{T}'$. Now

$$(a \times b, a \times c) = \{a\} \times (b, c).$$

It follows that the bases for $\mathcal{T}$ and $\mathcal{T}'$ are the same. Hence $\mathcal{T}$ and $\mathcal{T}'$ are the same.

Now let $\mathcal{T}''$ be the standard topology on $\mathbb{R}^2$. We show that $\mathcal{T}'$ is strictly finer than $\mathcal{T}''$. Consider the basis element $(a, b) \times (c, d)$ for $\mathcal{T}''$ and suppose $x \times y \in (a, b) \times (c, d)$. Then $\{x\} \times (c, d)$ is a basis element for $\mathcal{T}'$ and $x \times y \in \{x\} \times (c, d) \subset (a, b) \times (c, d)$. On the other hand, there is no basis element for $\mathcal{T}''$ that contains $\{x\} \times (c, d)$. ■

S17 Exercises

2. If A is closed in Y and Y is closed in X, then A is closed in X.
Proof. Since A is closed in Y, $A = Y \cap B$ for some set B closed in X. Since Y and B are both closed in X, so is $Y \cap B$. ∎

3. If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.
Proof. Suppose A is closed in X and B is closed in Y. Then $X - A$ is open in X and $Y - B$ is open in Y. Observe that

$$(X \times Y) - (A \times B) = ((X - A) \times Y) \cup (X \times (Y - B)).$$

Then $(X - A) \times Y$ is open in $X \times Y$ since $X - A$ is open in X and Y is open in Y. Similarly, $X \times (Y - B)$ is open in $X \times Y$. Therefore, their union $(X \times Y) - (A \times B)$ is open in $X \times Y$. ∎

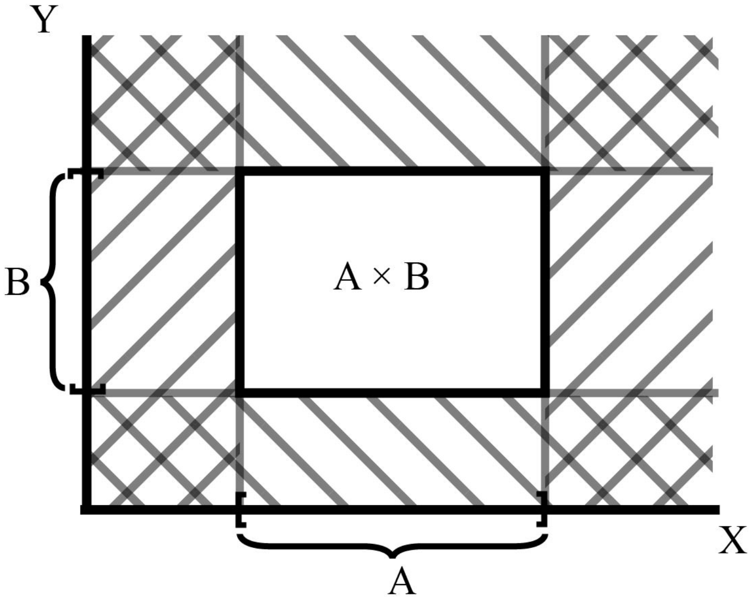

Figure 17.1

6. Let A, B, and A_α denote subsets of a space X. Then
(a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$..
(c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

Proof. **(a)** Suppose $A \subset B$. Let $x \in \bar{A}$. If U is a neighborhood of x, then U intersects A by Theorem 17.5. Since $A \subset B$, it follows that U intersects B. Thus $x \in \bar{B}$ by Theorem 17.5. It follows that $\bar{A} \subset \bar{B}$.

(b) Let $x \in \bar{A} \cup \bar{B}$. Then $x \in \bar{A}$ or $x \in \bar{B}$; say $x \in \bar{A}$. If U is a neighborhood of x, then U intersects A. Since $A \subset A \cup B$, it follows that U intersects $A \cup B$. Thus $x \in \overline{A \cup B}$. It follows that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

For the reverse inclusion, suppose $x \notin \bar{A} \cup \bar{B}$. Then $x \notin \bar{A}$ and $x \notin \bar{B}$. Since $x \notin \bar{A}$, there is a neighborhood U_1 of x that does not intersect A. Since $x \notin \bar{B}$, there is a neighborhood U_2 of x that does not intersect B. Then $U_1 \cap U_2$ is a neighborhood of x that does not intersect $A \cup B$. Thus $x \notin \overline{A \cup B}$. It follows that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

(c) By a generalization of the argument in the first paragraph of part (b), $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$. We now give an example where equality fails. Let $A_\alpha = \{1/\alpha\}$ for $\alpha \in \mathbb{Z}_+$. Then $\overline{\bigcup A_\alpha} = (\bigcup A_\alpha) \cup \{0\}$, but $\bigcup \bar{A}_\alpha = \bigcup A_\alpha$. ■

9. Let $A \subset X$ and $B \subset Y$. In the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Proof. Suppose $x \times y \in \bar{A} \times \bar{B}$. If $U \times V$ is a neighborhood of $x \times y$, then U intersects A and V intersects B by Theorem 17.5. Since

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V),$$

it follows that $U \times V$ intersects $A \times B$. Thus $x \times y \in \overline{A \times B}$ by Theorem 17.5. It follows that $\bar{A} \times \bar{B} \subset \overline{A \times B}$.

For the reverse inclusion, suppose $x \times y \in \overline{A \times B}$. If U is a neighborhood of x and V is a neighborhood of y, then $U \times V$ intersects $A \times B$ by Theorem 17.5. By the above equality, U intersects A and V intersects B. Thus $x \in \bar{A}$ and $y \in \bar{B}$, so that $x \times y \in \bar{A} \times \bar{B}$. It follows that $\overline{A \times B} \subset \bar{A} \times \bar{B}$. ■

15. The T_1 axiom is equivalent to the condition that for each pair of distinct points of X, each has a neighborhood not containing the other.

Proof. The statement "X satisfies the T_1 axiom" is equivalent to "one point sets in X are closed." This is in turn equivalent to "y is not a limit point of $\{x\}$ for each pair of distinct points x, y of X" by Corollary 17.7. This is in turn equivalent to "y has a neighborhood not containing x for each pair of distinct points x, y of X." This proves the result. ■

16. EXAMPLE. Consider the following five topologies on $\mathbb{R}$:

$$\begin{aligned} \mathcal{T}_1 &= \text{the standard topology,} \\ \mathcal{T}_2 &= \text{the topology of } \mathbb{R}_k, \\ \mathcal{T}_3 &= \text{the finite complement topology,} \\ \mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as a basis,} \\ \mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a) = \{x \mid x < a\} \text{ as basis.} \end{aligned}$$

(a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.

(b) Which of these topologies satisfy the Hausdorff axiom? The T_1 axiom?

Solution. (a)

$$\begin{aligned} \bar{K} &= K \cup \{0\} \text{ under } \mathcal{T}_1. \\ \bar{K} &= K \text{ under } \mathcal{T}_2 \text{ and } \mathcal{T}_4. \\ \bar{K} &= \mathbb{R} \text{ under } \mathcal{T}_3. \\ \bar{K} &= [0, \infty) \text{ under } \mathcal{T}_5. \end{aligned}$$

(b) Clearly, $\mathbb{R}$ satisfies the Hausdorff axiom under $\mathcal{T}_1$. Thus $\mathbb{R}$ satisfies the T_1 axiom under $\mathcal{T}_1$ by Theorem 17.8. Similarly, $\mathbb{R}$ satisfies Hausdorff axiom and the T_1 axiom under $\mathcal{T}_2$, and $\mathcal{T}_4$. $\mathbb{R}$ does not satisfy the Hausdorff axiom or the T_1 axiom under $\mathcal{T}_5$ since 1 is a limit point of $\{0\}$, so that $\{0\}$ is not closed. Finally, $\mathbb{R}$ satisfies the T_1 axiom under $\mathcal{T}_3$ since finite sets are closed. But $\mathbb{R}$ does not satisfy the Hausdorff axiom under $\mathcal{T}_3$ since any two open sets intersect each other. □

S18 Exercises

2. EXAMPLE. If $f\colon X \longrightarrow Y$ is continuous and x is a limit point of a subset A of X, then it is not necessarily true that $f(x)$ is a limit point of $f(A)$. For example, define $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = 1$ for all x and let $A = [0, 2]$. Then 1 is a limit point of A, but $f(1) = 1$ is not a limit point of $f(A) = \{1\}$.

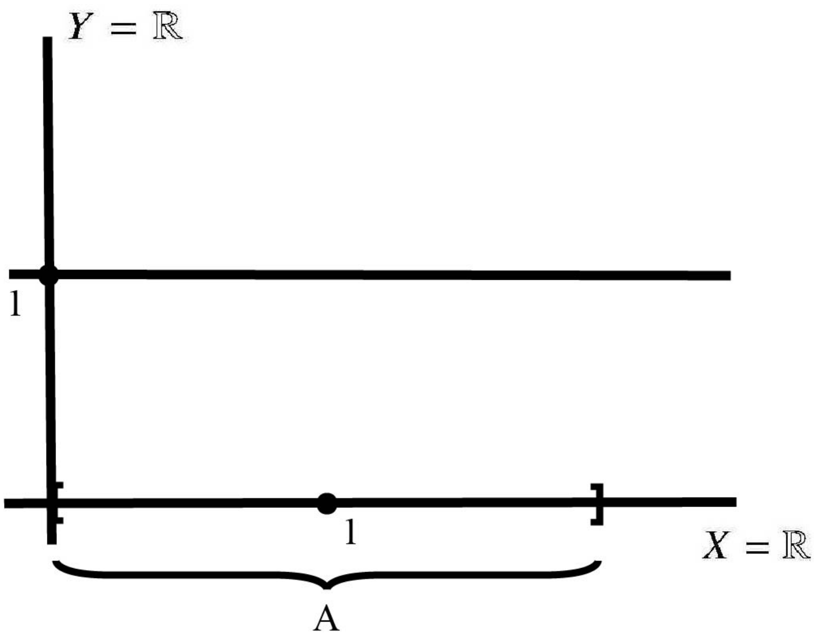

Figure 18.1

4. Given $x_0 \in X$ and $y_0 \in Y$, the maps $f\colon X \longrightarrow X \times Y$ and $g\colon Y \longrightarrow X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Proof. Let $h\colon X \longrightarrow X \times y_0$ be the function obtained by restricting the range of f to $X \times y_0$. Clearly, h is bijective. Observe that the open sets in $X \times y_0$ are the sets of the form $U \times y_0$, where U is open in X. Now

$$h(U) = U \times y_0.$$

Thus h is a homeomorphism, so that f is an imbedding. Similarly, g is an imbedding. ■

9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$; let $f\colon X \to Y$; suppose $f|A_\alpha$ is continuous for each α.

(a) If the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each set A_α is closed, but f is not continuous.

(c) An indexed family of sets $\{A_\alpha\}$ is said to be **locally finite** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. If the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Proof. **(a)** Let C be a closed subset of Y. Then

$$f^{-1}(C) = \bigcup_\alpha (f|A_\alpha)^{-1}(C).$$

Since each $(f|A_\alpha)^{-1}(C)$ is closed in A_α and each A_α is closed in X, it follows that each $(f|A_\alpha)^{-1}(C)$ is closed in X. Their union $f^{-1}(C)$ is thus closed in X.

(b) Let $A_0 = (-\infty, 0]$ and $A_\alpha = [1/\alpha, \infty)$ for $\alpha \in \mathbb{Z}_+$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x \le 0. \end{cases}$$

Then it is easy to see that $f|A_\alpha$ is continuous for each α, but f is not continuous.

(c) Let C be a closed subset of Y and let $B_\alpha = (f|A_\alpha)^{-1}(C)$ for each α. Suppose x is a limit point of $f^{-1}(C)$. Then there is a neighborhood of x that intersects B_α for only finitely many α; say for $\alpha \in J$. Then x is a limit point of $\bigcup_{\alpha\in J} B_\alpha$. Thus $x \in \bigcup_{\alpha\in J} B_\alpha \subset f^{-1}(C)$. ■

S19 Exercises

2. Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Proof. First, suppose $\prod X_\alpha$ is given the box topology. Let U_α be open in X_α for each α. Then a set of the form $(\prod A_\alpha) \cap (\prod U_\alpha)$ is a general basis element for the subspace topology on $\prod A_\alpha$ and a set of the form $\prod (A_\alpha \cap U_\alpha)$ is a general element for the box topology on $\prod A_\alpha$. Since

$$\prod (A_\alpha \cap U_\alpha) = (\prod A_\alpha) \cap (\prod U_\alpha),$$

it follows that the subspace topology on $\prod A_\alpha$ and the box topology on $\prod A_\alpha$ are the same.

Now suppose $\prod X_\alpha$ is given the product topology. Given any index β, let U_β be open in X_β. Then a set of the form $(\prod A_\alpha) \cap \pi_\beta^{-1}(U_\beta)$ is a general subbasis element for the subspace topology on $\prod A_\alpha$. If $V_\beta = A_\beta \cap U_\beta$ and π'_β is the function obtained by restricting the domain of π_β to $\prod A_\alpha$, then a set of the form $(\pi'_\beta)^{-1}(V_\beta)$ is a general subbasis element for the product topology on $\prod A_\alpha$. Now

$$(\pi'_\beta)^{-1}(V_\beta) = (\prod A_\alpha) \cap \pi_\beta^{-1}(U_\beta).$$

Thus the subspace topology on $\prod A_\alpha$ and the product topology on $\prod A_\alpha$ are the same. ■

3. If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

Proof. Suppose $\prod X_\alpha$ is given either the box or product topology. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points of $\prod X_\alpha$. Then there is some index β with $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there are disjoint open sets U and V in X_β containing x_β and y_β, respectively. Then $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively. See figure 19.1. ■

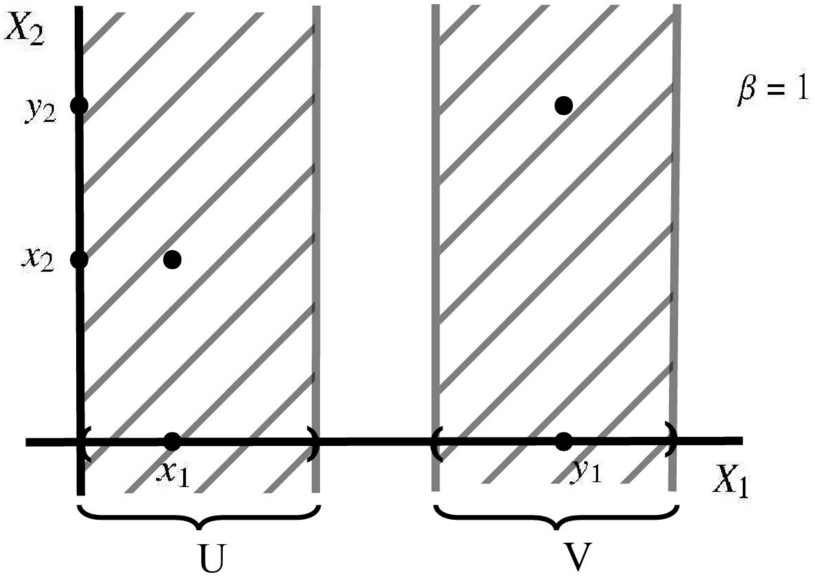

Figure 19.1

7. EXAMPLE. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. We find the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies.

The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\infty$ in the box topology. To see this, let $\mathbb{R}^\omega$ have the box topology and suppose $\mathbf{y} = (y_1, y_2, \ldots)$ does not belong to $\mathbb{R}^\infty$. Let $U = \prod U_\alpha$, where $U_\alpha = \mathbb{R}$ if $y_\alpha = 0$ and U_α is a neighborhood of y_α that does not include 0 if $y_\alpha \neq 0$. Then U is a basis element for $\mathbb{R}^\omega$ containing $\mathbf{y}$ that does not intersect $\mathbb{R}^\infty$.

The closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ is $\mathbb{R}^\omega$ in the product topology. To see this, let $\mathbb{R}^\omega$ have the product topology and suppose $\mathbf{y} = (y_1, y_2, \ldots)$ does not belong to $\mathbb{R}^\infty$. Let $U = \prod U_\alpha$ be a basis element for $\mathbb{R}^\omega$ containing $\mathbf{y}$. Let $\mathbf{x} = (x_1, x_2, \ldots)$, where $x_\alpha = 0$ if $U_\alpha = \mathbb{R}$ and $x_\alpha = y_\alpha$ otherwise. Then $\mathbf{x}$ belongs to both U and $\mathbb{R}^\infty$, so that U intersects $\mathbb{R}^\infty$.

S20 Exercises

2. EXAMPLE. We show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable. Define a metric d on $\mathbb{R} \times \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |y_2 - x_2| & \text{if } x_1 = y_1 \text{ and } |y_2 - x_2| < 1, \\ 1 & \text{otherwise.} \end{cases}$$

The properties for metric are satisfied trivially except for the triangle inequality:

$$d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}).$$

If all the first coordinates are equal, then d reduces to the standard bounded metric on $\mathbb{R}^2$, so that the inequality holds. If $x_1 \neq y_1$ or $y_1 \neq z_1$, then the right side of the inequality is at least 1 and the left side of the inequality is at most 1, so that the inequality holds. if $x_1 \neq z_1$, then $x_1 \neq y_1$ or $y_1 \neq z_1$.

We now show that the that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as that given by the metric d. Let $\mathbf{x} = x \times y$ and $\epsilon < 1$. Then the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ has as basis all sets of the form $(x \times (y - \epsilon), x \times (y + \epsilon))$ and the d - topology has as basis all sets of the form $B_d(\mathbf{x}, \epsilon)$. Since

$$B_d(\mathbf{x}, \epsilon) = (x \times (y - \epsilon), x \times (y + \epsilon)),$$

it follows that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and the d - topology are the same.

6. EXAMPLE. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

(a) $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
(b) $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
(c) We have

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

Solution. **(a)** Let $\mathbf{y} = (y_n)_{n \in \mathbb{Z}_+}$, where $y_n = x_n + n\epsilon/(n+1)$ for each n. Clearly, $\mathbf{y} \in U(\mathbf{x}, \epsilon)$. Since $\bar{d}(x_n, y_n) = n\epsilon/(n+1)$ for each n, it follows that

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\, n\epsilon/(n+1) \mid n \in \mathbb{Z}_+ \} = \epsilon.$$

Thus $\mathbf{y} \notin B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. Thus $U(\mathbf{x}, \epsilon) \neq B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
(b) Let $\mathbf{y} = (y_n)_{n \in \mathbb{Z}_+}$, where $y_n = x_n + n\epsilon/(n+1)$ for each n. If $U(\mathbf{x}, \epsilon)$ were open in the uniform topology, it would contain some δ-ball $B_{\bar{\rho}}(\mathbf{y}, \delta)$ centered at $\mathbf{y}$. But clearly, if $\mathbf{z} = (y_n + \delta/2)_{n \in \mathbb{Z}_+}$, then $\mathbf{z} \in B_{\bar{\rho}}(\mathbf{y}, \delta)$ and $\mathbf{z} \notin U(\mathbf{x}, \epsilon)$.
(c) Let $\mathbf{y} \in \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta)$. Then there is $\delta < \epsilon$ with $\mathbf{y} \in U(\mathbf{x}, \delta)$. Then $\bar{\rho}(\mathbf{x}, \mathbf{y}) \leq \delta < \epsilon$, so that $\mathbf{y} \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.

Conversely, suppose $\mathbf{y} \notin \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta)$. If $\delta < \epsilon$, then $\mathbf{y} \notin U(\mathbf{x}, \delta)$. Thus $\bar{\rho}(\mathbf{x}, \mathbf{y}) > \delta$. It follows that

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) \geq \sup\{\, \delta \mid \delta < \epsilon \} = \epsilon,$$

so that $\mathbf{y} \notin B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. □

S22 Exercises

3. EXAMPLE. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. We show that q is a quotient map that is neither open or closed.

First, we show that q is a quotient map. Clearly, q is surjective and continuous. Now let W be a saturated open subset of A and let $x \in q(W)$. Then $x \times 0 \in W$ and there is a basis element $(U \times V) \cap A$ for A such that $x \times 0 \in (U \times V) \cap A \subset W$. Hence $x \in U \subset q(W)$.

Next, we show that q is not open. The subset $U = [0, 1) \times (0, 1)$ of A is open in A, but $q(U) = [0, 1)$, which is not open in $\mathbb{R}$. Next, we show that q is not closed. The subset

$$C = \{x \times y \mid xy = 1\} \cap A$$

of A is open in A, but $q(C) = (0, \infty)$, which is not closed in $\mathbb{R}$.

4. EXAMPLE.

(a) Define and equivalence relation on the plane $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if} \quad x_0 + y_0^2 = x_1 + y_1^2.$$

Let X^* be the corresponding quotient space. We show that X^* is homeomorphic to $\mathbb{R}$.

The elements of X^* are the parabolas $\{x \times y \mid x + y^2 = c\}$, where $c \in \mathbb{R}$. Define a map $g : X \to \mathbb{R}$ by $g(x \times y) = x + y^2$; then g is surjective and continuous. The quotient space whose elements are the sets $g^{-1}(\{c\})$ is simply X^*. We show that g is a quotient map. It follows from Corollary 22.3 that g induces a homeomorphism $f : X^* \to \mathbb{R}$.

Let W be a saturated open subset of X and let $x \in g(W)$. Then $x \times 0 \in W$ and there is a basis element $U \times V$ for X such that $x \times 0 \in U \times V \subset W$. Then $x \in U \subset g(W)$.

(b) Define an equivalence relation on $X = \mathbb{R}^2$ as follows:

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if} \quad x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Let X^* be the corresponding quotient space. We show that X^* is homeomorphic to $\overline{\mathbb{R}}_+$.

The elements of X^* are the circles $\{x \times y \mid x^2 + y^2 = c\}$, where $c \in \mathbb{R}_+$, along with the one point set $\{0 \times 0\}$. Define a map $g : X \to \overline{\mathbb{R}}_+$ by $g(x \times y) = \sqrt{x^2 + y^2}$; then g is surjective and continuous.The quotient space whose elements are the sets $g^{-1}(\{c\})$, where $c \in \overline{\mathbb{R}}_+$ is simply X^*. By an analogous argument to the argument in part (a), g is a quotient map. Thus g induces a homeomorphism $f : X \to \overline{\mathbb{R}}_+$.

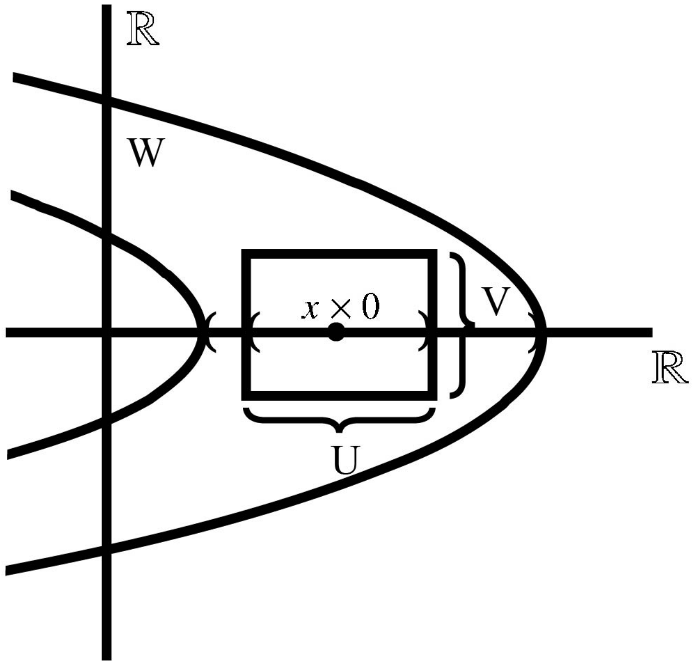

Figure 22.1

S23 Exercises

2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Then $\bigcup A_n$ is connected.
Proof. Let $A = \bigcup A_n$. Assume, for contradiction, that $A = B \cup C$ is a separation of A. Since A_1 is connected, it follows that A_1 must lie entirely in one of the sets B or C; say $A_1 \subset B$. Now suppose $A_n \subset B$. Since $A_n \cap A_{n+1} \neq \emptyset$, it follows that $B \cap A_{n+1} \neq \emptyset$. Thus $A_{n+1} \subset B$. Therefore, $A_n \subset B$ for all n by induction. Thus $A \subset B$. This is a contradiction since C is nonempty. ■

7. EXAMPLE. The space $\mathbb{R}_l$ is not connected. To see this, observe that for any $x \in \mathbb{R}$, we have

$$(-\infty, x) = \bigcup_{n \in \mathbb{Z}_+} [x - n, x) \quad \text{and} \quad [x, \infty) = \bigcup_{n \in \mathbb{Z}_+} [x, x + n).$$

Thus $(-\infty, x)$ and $[x, \infty)$ are open in $\mathbb{R}_l$. Hence, $(-\infty, x) \cup [x, \infty)$ is a separation of R_l.

9. Let A be a proper subset of X and let B be a proper subset of Y. If X and Y are connected, then

$$(X \times Y) - (A \times B)$$

is connected.
Proof. Choose a "base point" $a \times b \in (X - A) \times (Y - B)$. Then each "T-shaped" space $T_{xy} = (X \times y) \cup (x \times Y)$ is connected since it is the union of connected spaces that have the point $x \times y$ in common. Thus each union $\bigcup_{x \in X - A} T_{xy}$ is connected since it is the union of a collection of connected spaces that have the point $a \times y$ in common. Thus

$$(X \times Y) - (A \times B) = \bigcup_{y \in Y - B} \bigcup_{x \in X - A} T_{xy}$$

is connected since it is the union of a collection of connected spaces that have the point $a \times b$ in common. ■

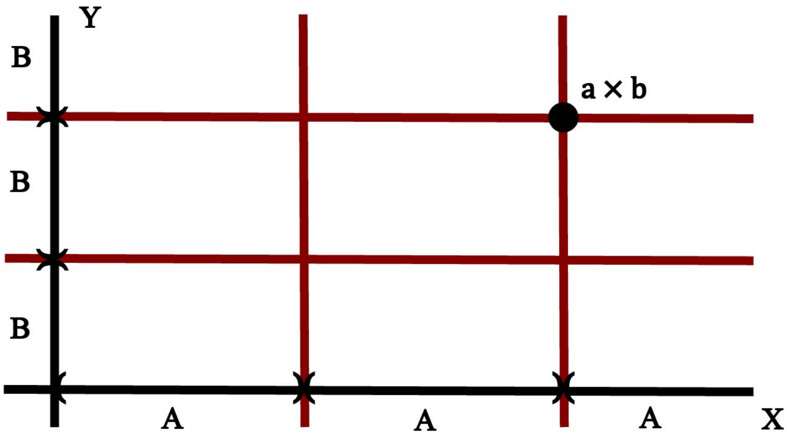

Figure 23.1

11. Let $p\colon X \longrightarrow Y$ be a quotient map. If each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Proof. Suppose $X = U \cup V$ is a separation of X. Let $A = p(U)$ and $B = p(V)$. Clearly, A and B are nonempty and $A \cup B = Y$. Since each $p^{-1}(\{y\})$ is connected, it follows that $U = p^{-1}(A)$ and $V = p^{-1}(B)$. Thus A and B are open since p is a quotient map. Finally, A and B are disjoint since $U \cap V = p^{-1}(A \cap B)$. Thus $A \cup B$ forms a separation of Y, contradicting the assumption that Y is connected. ■

S24 Exercises

3. EXAMPLE. Let $f: X \longrightarrow X$ be continuous. If $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x of X is called a **fixed point** of f. What happens if X equals $[0, 1)$ or $(0, 1)$?

Solution. First, suppose $X = [0, 1]$. Then $g: X \longrightarrow \mathbb{R}$ given by $g(x) = f(x) - x$ is continuous. Clearly, $g(0) \geq 0$ and $g(1) \leq 0$. Then by the Intermediate value Theorem, there is a point $c \in X$ such that $g(c) = 0$. Since $g(c) = 0$, it follows that $f(c) = c$.

Now suppose $f: [0, 1) \longrightarrow [0, 1)$ is given by $f(x) = x/2 + 1/2$. Then f is continuous. However, $f(x) > x$ on $[0, 1)$. Now suppose $h: (0, 1) \longrightarrow (0, 1)$ is obtained by restricting the domain of f. Then h is continuous, but $h(x) > x$ on $(0, 1)$. □

5. EXAMPLE. We show which of the following sets in the dictionary order are linear continua:

(a) $\mathbb{Z}_+ \times [0, 1)$
(b) $[0, 1) \times \mathbb{Z}_+$
(c) $[0, 1) \times [0, 1]$
(d) $[0, 1] \times [0, 1)$

Even though $[0, 1)$ has subsets that are unbounded in $[0, 1)$, every element of $\mathbb{Z}_+$ has an immediate successor. Thus $\mathbb{Z}_+ \times [0, 1)$ has the least upper bound property. Clearly, $\mathbb{Z}_+ \times [0, 1)$ satisfies property (2) for linear continua. Thus $\mathbb{Z}_+ \times [0, 1)$ is a linear continuum.

Since every subset of $[0, 1]$ is bounded above in $[0, 1]$, it follows that $[0, 1) \times [0, 1]$ has the least upper bound property. Clearly, $[0, 1) \times [0, 1]$ satisfies property (2) for linear continuua.

Since $\mathbb{Z}_+$ has subsets that are unbounded, it follows that $[0, 1) \times \mathbb{Z}_+$ does not have the least upper bound property. Thus $[0, 1) \times \mathbb{Z}_+$ is not a linear continuum. Similarly, $[0, 1] \times [0, 1)$ is not a linear continuum.

9. EXAMPLE. Assume that $\mathbb{R}$ is uncountable. If A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 - A$ is path connected.

Solution. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2 - A$. It is easy to see that there are uncountably many lines passing through $\mathbf{x}$ that don't intersect A and uncountably many lines passing through $\mathbf{y}$ that don't intersect A. Choose one of the lines through $\mathbf{x}$ and one of the lines through $\mathbf{y}$ that do not intersect A and that intersect each other; say at $\mathbf{z}$. Then the broken-line path from $\mathbf{x}$ to $\mathbf{z}$ and then from $\mathbf{z}$ to $\mathbf{y}$ is a path in $\mathbb{R}^2 - A$ that joins $\mathbf{x}$ and $\mathbf{y}$. □

10. EXAMPLE. If U is an open connected subspace of $\mathbb{R}^2$, then U is path connected.

Solution. Given $\mathbf{x}_0 \in U$, let V be the set of all points in U that can be joined to $\mathbf{x}_0$ by a path in U. We show that V is both open and closed in U. This implies $V = U$, so that U is path connected.

We first show that V is open in U. Let $\mathbf{y} \in V$ and suppose $B(\mathbf{y}, \epsilon) \subset U$. Let $\mathbf{z} \in B(\mathbf{y}, \epsilon)$. Since $\mathbf{y} \in V$, there is a path in U from $\mathbf{x}_0$ to $\mathbf{y}$. Since ϵ-balls are path connected, there is a path in U from $\mathbf{y}$ to $\mathbf{z}$. Thus there is a path is U from $\mathbf{x}_0$ to $\mathbf{z}$, so that $\mathbf{z} \in V$. Thus $B(\mathbf{y}, \epsilon) \subset V$.

We next show that V is closed in U. Let $\mathbf{z}$ be a limit point of V in U and suppose $B(\mathbf{z}, \epsilon) \subset U$. Then $B(\mathbf{z}, \epsilon)$ intersects V at some point $\mathbf{y}$. Since $\mathbf{y} \in V$, there is a path in U from $\mathbf{x}_0$ to $\mathbf{y}$. Since ϵ-balls are path connected, there is a path in U from $\mathbf{y}$ to $\mathbf{z}$. Thus there is a path in U from $\mathbf{x}_0$ to $\mathbf{z}$, so that $\mathbf{z} \in V$. □

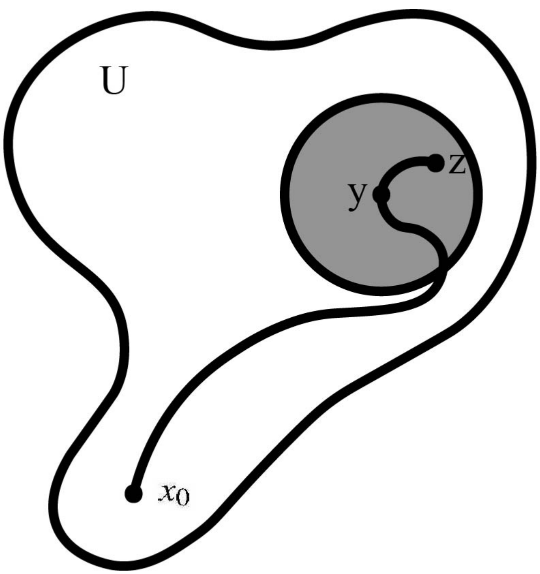

Figure 24.1

S26 Exercises

1.

(a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose $\mathcal{T}' \supset \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) If X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

Proof. **(a)** If X is compact under $\mathcal{T}'$, then X is compact under $\mathcal{T}$. To see this, suppose X is compact under $\mathcal{T}'$. Let $\mathcal{A}$ be an open covering of X under $\mathcal{T}$. Then $\mathcal{A}$ is an open covering of X under $\mathcal{T}'$, so that $\mathcal{A}$ contains a finite subcollection covering X. Hence, X is compact under $\mathcal{T}$.

However, the converse is not true. To see this, consider the following subset of $\mathbb{R}$:

$$X = \{-1/n \mid n \in \mathbb{Z}_+\} \cup \{0\}.$$

Then X is a compact subspace of $\mathbb{R}$. But X is not a compact subspace of $\mathbb{R}_k$ since

$$\mathcal{A} = \{[-1/n, -1/(n+1)) \mid n \in \mathbb{Z}_+\} \cup \{[0, 1)\}$$

is a covering of X by sets open in $\mathbb{R}_k$ that has no finite subcollection covering X.

(b) Suppose $\mathcal{T}$ and $\mathcal{T}'$ are comparable; say $\mathcal{T}' \supset \mathcal{T}$. We show that $\mathcal{T}' = \mathcal{T}$ Let A be closed in X under $\mathcal{T}'$. Then A is a compact subspace of X under $\mathcal{T}'$ by Theorem 26.2. Thus A is a compact subspace of X under $\mathcal{T}$ by part (a). Hence A is closed in X under $\mathcal{T}$ by Theorem 26.3. It follows that $\mathcal{T}' \subset \mathcal{T}$. ■

2.

(a) In the finite complement topology on $\mathbb{R}$, every subspace is compact.

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} - A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

Proof. **(a)** Let $A \subset \mathbb{R}$ and suppose $\mathcal{A}$ is an open covering of A. Let U be a nonempty open set in $\mathcal{A}$. Then U contains all but finitely many points of A. For each a not in U, choose an open set U_a in $\mathcal{A}$ containing a. Then

$$\{U\} \cup \{U_a \mid a \in A - U\}$$

is a finite subcollection of $\mathcal{A}$ covering A.

(b) The space $[0, 1]$ is not compact. To see this, let $K = \{1/n \mid n \in \mathbb{Z}_+\}$. Then

$$\mathcal{A} = \{([0, 1] - K) \cup \{1/n\} \mid n \in \mathbb{Z}_+\}$$

is an open covering of $[0, 1]$ that has no finite subcollection covering $[0, 1]$. ■

5. Let A and B be disjoint compact subspaces of the Hausdorff space X. Then there exist disjoint open sets U and V containing A and B, respectively.

Proof. For each point a of A, choose disjoint open sets U_a and V_a containing a and B, respectively. The existence of such open sets is guaranteed by Lemma 26.4. The collection $\{U_a \mid a \in A\}$ is a covering of A by sets open in X. Therefore, finitely many of them $U_{a_1}, \ldots, U_{a_n}$ cover A. Let

$$U = U_{a_1} \cup \cdots \cup U_{a_n}$$

and

$$V = V_{a_1} \cap \cdots \cap V_{a_n}.$$

Then U and V are disjoint open sets containing A and B, respectively. ■

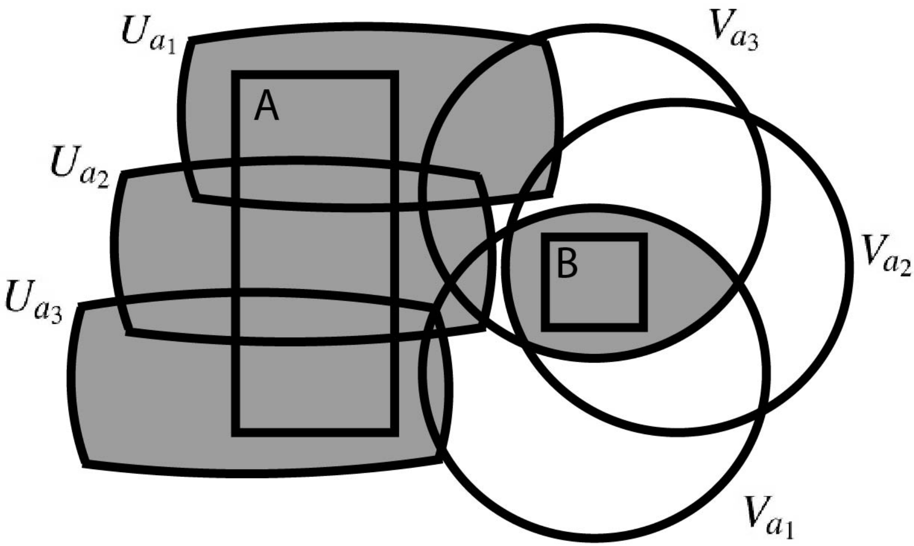

Figure 26.1

11. Theorem. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

Proof. Suppose $Y = C \cup D$ is a separation of Y. Then C and D are compact, so that there are disjoint open sets U and V in X containing C and D, respectively, by Exercise 26.5. Now consider the collection

$$\mathcal{C} = \{A - (U \cup V) \mid A \in \mathcal{A}\}.$$

Then $\mathcal{C}$ is simply ordered under proper set inclusion. Since each $C \in \mathcal{C}$ is nonempty (if $C \in \mathcal{C}$ were empty, then $A \subset U \cup V$, so that $U \cap A$ and $V \cap A$ forms a separation of A), it follows that $\mathcal{C}$ has the finite intersection property. But this means that

$$Y - (U \cup V) = \bigcap_{C \in \mathcal{C}} C$$

is nonempty (by Theorem 26.9), which contradicts the fact that $Y \subset U \cup V$. ■

S28 Exercises

3. Let X be limit point compact.
(a) If $f: X \to Y$ is continuous, does it follow that $f(x)$ is limit point compact?
(b) If A is a closed subset of X, does it follow that A is limit point compact?
(c) If X is a subspace of a Hausdorff space Z, does it follow that X is closed in Z?
Proof.(a) If f is continuous, it does not necessarily follow that $f(X)$ is limit point compact. For example, consider $A = \{0, 1\}$ in the indiscrete topology and let $X = \mathbb{Z}_+ \times A$. Then X is limit point compact by Example 1. Define $f: \mathbb{Z}_+ \times A \to \mathbb{Z}_+$ by $f(b \times a) = b$. Then f is continuous since if $\{b\}$ is any basis element for $\mathbb{Z}_+$, then $f^{-1}(\{b\}) = \{b\} \times A$ is open in $\mathbb{Z}_+ \times A$. However, $f(X) = \mathbb{Z}_+$ is not limit point compact since $\mathbb{Z}_+$ has no limit point.
(b) If A is a closed subset of X, then A is limit point compact. To see this, let B be an infinite subset of A. Then B has a limit point b in X. Since $B \subset A$, it follows that b is a limit point of A in X. Since A is closed, it bollows that $b \in A$. It is easy to see that b is a limit point of B in A.
(c) If X is a subspace of a Hausdorff space Z, it does not necessarily follow that X is closed in Z. For example, let $Z = \bar{S}_\Omega$ and $X = S_\Omega$. Then X is limit point compact by Example 2. Further, Z is Hausdorff since Z is a simply ordered set in the order topology. But X is not closed in Z since Ω is a limit point of X and $\Omega \notin X$. For assume Ω is not a limit point of X. Then there is $\alpha < \Omega$ such that $(\alpha, \Omega) = \emptyset$. But S_α is countable, so that $S_\Omega = S_\alpha \cup \{\alpha\}$ is countable, which is a contradiction. ■

4. A space X is said to be **countably compact** if every countable open covering of X contains a finite subcollection that covers X. For a T_1 space, countable compactness is equivalent to limit point compactness.

Proof. Suppose X is countably compact. Let A be a countable subset of X. We show that if A has no limit points, then A is finite. So suppose A has no limit points. Then A is closed. Further, for each $a \in A$, there is a neighborhood U_a of a that intersects A at the point a alone. Then X is covered by the open set $X - A$ and the open sets U_a. Thus X can be covered by finitely many of these sets. Since $X - A$ does not intersect A, and each U_a contains only one point of A, the set A must be finite.

Now suppose A is an infinite subset of X. Then A has an infinitely countable subset B. By the preceding paragraph, B has a limit point b. Then b is a limit point of A. Thus X is limit point compact.

For the converse, suppose X is not countably compact. Then there is a countable open covering $\{U_n \mid n \in \mathbb{Z}_+\}$ of X such that no finite subcollection covers X. For each positive integer k, choose $x_k \notin U_1 \cup \cdots \cup U_k$. Let

$$A = \{x_n \mid n \in \mathbb{Z}_+\}.$$

Now suppose $x \in X$ and let k be the least positive integer for which $x \in U_k$. Then U_k is a neighborhood of x that contains at most $k - 1$ points of A. Since X is a T_1 space, it follows that x is not a limit point of A. It follows that A has no limit points, so that X is not limit point compact. ■

S29 Exercises

5. If $f\colon X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, then f extends to a homeomorphism of their one-point compactifications.

Proof. Let $Y_1 = X_1 \cup \{\infty_1\}$ and $Y_2 = X_2 \cup \{\infty_2\}$ be the one-point compactifications of X_1 and X_2, respectively. Define $h\colon Y_1 \to Y_2$ by letting $h(x) = f(x)$ for x in X_1 and letting $h(\infty_1) = \infty_2$. We show that if U is open in Y_1, then $h(U)$ is open in Y_2. Symmetry implies that h is a homeomorphism.

First, consider the case where U does not contain ∞_1. Then U is open in X_1. Observe that $h(U) = f(U)$ is open in X_2 since f is a homeomorphism. Thus $h(U)$ is open in Y_2.

Next, suppose U contains ∞_1. Then $U = Y_1 - C$, where C is a compact subspace of X_1. Observe that

$$h(U) = Y_2 - f(C)$$

Since f is a homeomorphism, $f(C)$ is a compact subspace of X_2 by Theorem 26.5. Thus $h(U)$ is open in Y_2. ■

8. EXAMPLE. We show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic to the subspace $A = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$. Let $Y = \mathbb{Z}_+ \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}_+$.

Every subset of $\{1/n \mid n \in \mathbb{Z}_+\}$ is open in A. Suppose U is an open set in A containing 0. Let $B = (a, b) \cap A$ be a basis element for A such that $0 \in (a, b) \cap A \subset U$. Let n be the least natural number such that $1/n \in B$. Then

$$U = A - C$$

where $C \subset \{1/m \mid m \in \mathbb{Z}_+ \text{ and } m < n\}$. Thus the open sets in A containing 0 are the sets of the form $A - C$, where C is a finite subset of $\{1/n \mid n \in \mathbb{Z}_+\}$.

Every subset of $\mathbb{Z}_+$ is open in Y. Since the compact subspaces of $\mathbb{Z}_+$ are the finite sets in $\mathbb{Z}_+$, the open sets in Y containing ∞ are the sets of the form $Y - C$, where C is a finite subset of $\mathbb{Z}_+$.

Now define $f\colon A \to Y$ by letting $f(1/n) = n$ if $n \in \mathbb{Z}_+$ and letting $f(0) = \infty$. Then it is easy to see that f is a homeomorphism.

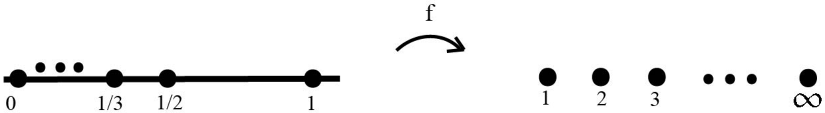

Figure 29.1

4. EXAMPLE. The space $[0,1]^\omega$ is not locally compact in the uniform topology; no ϵ-ball with $\epsilon < 1$ centered at 0 is contained in a compact subspace of $[0,1]^\omega$. To see this, let $B = B_{\bar{\rho}}(0,\epsilon)$ where $\epsilon < 1$. If B were contained in a compact subspace, then its closure $\bar{B} = [0,\epsilon]^\omega$ would be compact, which it is not. To see that $[0,\epsilon]^\omega$ is not compact, let $X = \{\epsilon/3, 2\epsilon/3\}$. Then $\{B_{\bar{\rho}}(\mathbf{x},\epsilon/2) \mid \mathbf{x} \in X^\omega\}$ is an open covering of $[0,\epsilon]^\omega$ that has no finite subcollection covering $[0,\epsilon]^\omega$.

S30 Exercises

3. Let X have a countabe basis; let A be an uncountable subset of X. Then uncountably many points of A are limit points of A.
Proof. We prove the contrapositive: If countably many points of A are limit points of A, then A is countable. Let A' be the set of limit points of A and let $\mathcal{B}$ be a countable basis for X. For each $a \in A - A'$, choose an element B_a of $\mathcal{B}$ that intersects A at the point a alone. Then the map $a \to B_a$ is an injection of $A - A'$ into $\mathcal{B}$, so that $A - A'$ is countable. Since A' is countable, so is A. ■

5.
(a) Every metrizable space with a countable dense subset has a countable basis.
(b) Every metrizable Lindelöf space has a countable basis.
Proof. **(a)** Let X be a metric space and suppose A is a countable dense subset of X. Let

$$\mathcal{B} = \{B(a, \delta) \mid a \in A, \delta \in \mathbb{Q}_+\}.$$

Then $\mathcal{B}$ is countable since A and $\mathbb{Q}_+$ are countable. Further, $\mathcal{B}$ is a basis for X: If $B(x, \epsilon)$ is an epsilon-ball centered at x, choose $\delta \in \mathbb{Q}_+$ with $\delta < \epsilon/2$. since A is dense, there is $a \in A$ with $a \in B(x, \delta)$. Then $x \in B(a, \delta) \subset B(x, \epsilon)$.
(b) Let X be a metrizable Lindelöf space. For each $\delta \in \mathbb{Q}_+$, the set $\{B(x, \delta) \mid x \in X\}$ has a countable subcollection $\mathcal{B}_\delta$ covering X. Then $\mathcal{B} = \bigcup_{\delta \in \mathbb{Q}_+} \mathcal{B}_\delta$ is a countable. Further, $\mathcal{B}$ is a basis for X: If $B(x, \epsilon)$ is an ϵ-ball centered at x, choose $\delta \in \mathbb{Q}_+$ with $\delta < \epsilon/2$. Since $\mathcal{B}_\delta$ covers X, there is $B(y, \delta) \in \mathcal{B}_\delta$ containing x. Then $x \in B(y, \delta) \subset B(x, \epsilon)$. ■

10. If X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.
Proof. Suppose $X = \prod X_\alpha$. Then each X_α has a countable dense subset A_α. We show that $\prod A_\alpha$ is a countable dense subset of X: It is countable since it is the countable product of countable sets. Further, if $(x_\alpha) \in X$, then $x_\alpha \in \bar{A}_\alpha$ by the denseness of each A_α. Thus $(x_\alpha) \in \prod \bar{A}_\alpha = \overline{\prod A_\alpha}$. It follows that $\overline{\prod A_\alpha} = X$. ■

S31 Exercises

1. If X is regular, every pair of points of X have neighborhoods whose closures are disjoint.
Proof. Let x_1 and x_2 be a pair of points of X. Using regularity, choose disjoint neighborohods U_1 and U_2 of x_1 and x_2, respectively. By Lemma 31.1, there is a neighborhhod V_1 of x_1 such that $\bar{V}_1 \subset U_1$. Similarly, there is a neighborhood V_2 of x_2 such that $\bar{V}_2 \subset U_2$. Then $\bar{V}_1$ and $\bar{V}_2$ are disjoint. ■

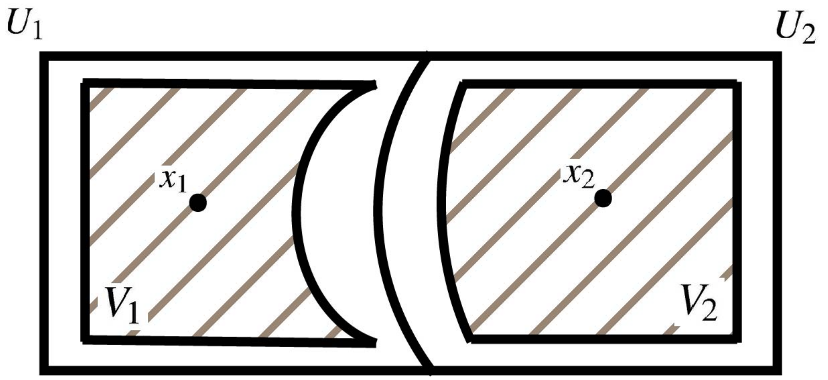

Figure 31.1

4. Let X and X' denote a single set under the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; assume $\mathcal{T}' \supset \mathcal{T}$. Clearly, X is Hausdorff implies X' is Hausdorff. But X is regular (or normal) does not imply X' is regular (or normal). For example, let $X = \mathbb{R}$ and $X' = \mathbb{R}_k$. Finally, X' is Hausdorff (or regular, or normal) does not imply X is Hausdorff (or regular, or normal). For example, let X be $\{0, 1\}$ in the indiscrete topology and let X' be $\{0, 1\}$ in the discrete topology.

S32 Exercises

4. Every regular Lindelöf space is normal.

Proof. This proof is analogous to the proof of Theorem 32.1. Let X be a regular Lindelöf space. Let A and B be disjoint closed subsets of X. Suppose $x \in A$. Let U be a neighborhood of x not intersecting B. Using regularity, there is a neighborhood V of x such that $\bar{V} \subset U$. By choosing such a neighborhood for each $x \in A$, we obtain a covering $\mathcal{A}$ of A by open sets whose closures do not intersect B. Since X is Lindelöf, there is a countable subcollection of $\mathcal{A}$ covering A. Thus we can index this subcollection with the positive integers; let us denote it by $\{U_n\}$.

Similarly, there is a countable collection $\{V_n\}$ of open sets covering B whose closures do not intersect A. Given n, define

$$U'_n = U_n - \bigcup_{i=1}^{n} \bar{V}_i \quad \text{and} \quad V'_n = V_n - \bigcup_{i=1}^{n} \bar{U}_i.$$

Let

$$U' = \bigcup_{n \in \mathbb{Z}_+} U'_n \quad \text{and} \quad V' = \bigcup_{n \in \mathbb{Z}_+} V'_n.$$

We show that U' and V' are disjoint sets containing A and B, respectively. Clearly, U' and V' are open. It is easy to see that $\{U'_n\}$ covers A. Thus $A \subset U'$. Similarly, $B \subset V'$. Finally, the sets U' and $V'$are disjoint. For if $x \in U' \cap V'$, then $x \in U'_j$ for some j and $x \in V'_k$ for some k. If $j \leq k$, then $x \in U_j$ and $x \notin \bar{U}_j$, which is a contradiction. A similar contradiction happens if $k < j$. ∎