4.3 Lemma Let $L(T')$ be an extension of $L(T)$. Suppose $\mathcal{A}'$ is a model of T' and $\mathcal{A}$ is the restriction of $\mathcal{A}'$ to $L(T)$. Then $\mathcal{A}(A) = \mathcal{A}'(A)$ for any closed formula A of $L(T)(\mathcal{A})$.

Proof. We may assume the same names of individuals of $\mathcal{A}$ are used for the names of individuals of $\mathcal{A}'$. Then this follows naturally from the fact that $\mathcal{A}(\mathbf{i}) = \mathcal{A}'(\mathbf{i})$ and $\mathbf{f}_{\mathcal{A}}$ is $\mathbf{f}_{\mathcal{A}'}$ and $\mathbf{p}_{\mathcal{A}}$ is $\mathbf{p}_{\mathcal{A}'}$ for all names $\mathbf{i}$, function symbols $\mathbf{f}$, and predicate symbols $\mathbf{p}$ of $L(T)(\mathcal{A})$, respectively.

First let a be a term. We show that $\mathcal{A}(a)$ is $\mathcal{A}'(a)$. /if $\mathbf{a}$ is a name, then $\mathcal{A}(\mathbf{a})$ is $\mathcal{A}'(\mathbf{a})$. If $\mathbf{a}$ is not a name, then $\mathbf{a}$ is $\mathbf{f}\mathbf{a}_1 \cdots \mathbf{a}_n$ so that $\mathcal{A}(\mathbf{a})$ is $\mathbf{f}_{\mathcal{A}}(\mathcal{A}(\mathbf{a}_1), \cdots, \mathcal{A}(\mathbf{a}_n))$, which is $\mathbf{f}_{\mathcal{A}'}(\mathcal{A}'(\mathbf{a}_1), \cdots, \mathcal{A}'(\mathbf{a}_n))$, which is $\mathcal{A}'(\mathbf{a})$.

Now suppose A is a formula. Suppose A is $a = b$. Then

$$\mathcal{A}(A) = \mathbf{T} \text{ iff } \mathcal{A}(a) = \mathcal{A}(b) \text{ iff } \mathcal{A}'(a) = \mathcal{A}'(b) \text{ iff } \mathcal{A}'(A) = \mathbf{T}.$$

Now suppose A is $\mathbf{p}\mathbf{a}_1 \cdots \mathbf{a}_n$, where $\mathbf{p}$ is not $=$. Then

$$\mathcal{A}(A) = \mathbf{T} \text{ iff } \mathbf{p}_{\mathcal{A}}(\mathcal{A}(\mathbf{a}_1), \ldots, \mathcal{A}(\mathbf{a}_n)) \text{ iff } \mathbf{p}_{\mathcal{A}'}(\mathcal{A}'(\mathbf{a}_1), \ldots, \mathcal{A}'(A)(\mathbf{a}_n)) \text{ iff } \mathcal{A}' = \mathbf{T}.$$

Now suppose A is $\neg B$. Then $\mathcal{A}(A) = H_\vee(\mathcal{A}(B)) = H_\vee(\mathcal{A}'(B)) = \mathcal{A}'(A)$. Now suppose A is $B \vee C$. Then $\mathcal{A}(A) = H_\vee(\mathcal{A}(B), \mathcal{A}(C))$. Now suppose A is $\exists \mathbf{x} B$. Then $\mathcal{A}(A) = T$ iff $\mathcal{A}(B_{\mathbf{x}}[\mathbf{i}])$ for some $\mathbf{i}$ iff $\mathcal{A}'(B_{\mathbf{x}}[\mathbf{i}])$ for some $\mathbf{i}$ iff $\mathcal{A}'(A) = T$.